Volunteer's Dilemma: Cost-Sharing Revisited

Rabah Amir[*], Jingwen Tian[†]

November 22, 2023

Abstract

We revisit the sociological dilemma of volunteering, namely that in the production of a collective good (e.g., rescuing the victim of a public crime) a dilemma is often posed to bystanders given the chance of free-riding as to who should be the volunteer to contribute, which usually results in a failure of production and thus social inefficiency. Past theory literature suggests that when the bystander group becomes larger both the individual's incentive of volunteering and the collective probability of the good's production should decrease. But the latter claim goes against intuition and is less grounded in view of the experiment results. We propose a cost-sharing model that allows individual's volunteering cost to decrease exponentially in the number of volunteers showing up, and show that the overall probability of production may increase in the number of players for sufficiently low volunteering cost, thus giving an alternative account of the reality and an explanation of the experiment results, that is, a larger group does erode individual's incentive to volunteer but with more bystanders it eventually favors the production of the collective good. Our result also implies that the dilemma is sensitive to the way in which the volunteering cost is modeled, which thus needs to be handled with caution.

[*]Department of Economics, the University of Iowa, Iowa City, IA, USA. Email: rabah-amir@uiowa.edu.
[†]Department of Economics, the University of Iowa, Iowa City, IA, USA. Email: gin-tian@uiowa.edu.

1 Introduction

The volunteer's dilemma (Diekmann, 1985) is a n-player game revolving around the production of a collective good, where each individual incurs cost c to volunteer, in exchange of benefit b once the good is produced, which requires only one player to volunteer. In general $b > c$, so an individual would have the incentive to volunteer given he is the only player in the game. But when there are other players, individual's incentive to volunteer is eroded by the hope that someone else will step out to do the job so that he can gain in free, i.e., by the chance of free-riding. This incentive to free-ride may produce very unexpected (and brutal) outcomes that are socially inefficient. A frequently referred to story in this context, which happened nearly sixty years ago, still shocking, is about a lady named Kitty Genovese living in the neighborhood of Queens in New York City, who was publicly assaulted and murdered in the presence of 38 witnesses who either saw or heard the attack, yet none of whom called the police. A later coverage by *The New York Times* called public attention to the problem of the *bystander effect*. However, as Weesie (1993) wisely commented, it is not the immorality or indifference of the witnesses that killed Ms. Genovese, it fact, the witnesses could be perfectly moral yet rational free-riders, but she is the victim of nature (chance) where all witnesses happened to assume someone else would have called the police. Then a natural question is, Would Kitty Genovese be luckier if there are only 5 witnesses, or if there are 500 witnesses? In other words, would a larger group favor or disfavor the production of the collective good, when there is reduced individual's incentive but increased chance of *someone* stepping out?

In Kitty Genovese's case, it is hard to imagine that she will be less likely to be saved if there are 500 witnesses than when there are 38 witnesses, which only happens when every one of them abstains from taking any action; How likely is that to take place? If it does, what kind of society it alarms of? That is, we think it is a more accurate description of reality that the victim will be more likely to be saved when there is a larger crowd of witnesses. Moreover, the experiment results reported in Weesie and Franzen (1998) do not provide any grounding for the speculation that the production probability (i.e., the probability of the collective good being produced) should decrease in the number of players; in fact, some of the results point to the opposite (see Table 4 in Weesie and Franzen, 1998). However, Diekmann's (1985) original model, though artfully revealing the rationale and inefficiency behind the volunteer's dilemma, gives way to the prediction that the production probability does decrease in the number of players, unless people are superrational, which means people would assume, in line with Kant's catogorical imperative, he and everyone else's actions are bound by the exact same morality (a universal law), so that when one player chooses his strategy, he supposes all other players will choose the same. This borrowed philosophical notion is in some sense too strong, goes against the modern Economics rational-man assumption and against the whole spirit of game theory.

Following Diekmann (1985), Weesie and his coauthor generalized the original model in several directions in order to see how the characteristics of the dilemma respond to these generalizations. Weesie (1993) proposed an alternative *timing* game to Diekmann's static one, where players make the decision of *when* he will step out to contribute, given other players have not done so, and showed that the asymmetric version (in terms of players' cost-to-gain ratio) of this timing game solves the dilemma in the sense that the most favorable player will volunteer immediately. Weesie (1994) further shows that the observability of the behaviors of other players (manifested as the timing game) and the uncertainty with respect to the payoffs of other players both enhance the likelihood that a player volunteers and the production probability, except when behaviors are unobservable and the uncertainty is high, a larger group may contrarily favor the good's production for *sufficiently* large n. Lastly, Weesie and Franzen (1998) brought in the idea of cost-sharing, modified the volunteering cost to be divisible among the volunteers either by evenly splitting (i.e., each volunteer incurs cost $\frac{c}{k}$ when k players step out to volunteer) or by a lottery which selects one volunteer to carry the full burden of cost. They concluded using theoretical analysis that an individual is less likely to volunteer in larger groups and a larger group is less likely to provide the collective good, but their experiment results only partially support the former conclusion while completely refutes the later.

Therefore, except by resorting to the intricate philosophical notion of superrationality or by bringing in incomplete information (and only for sufficiently large n), to our knowledge, there is yet no proper model proposed to account for the possibility that the collective good's production probability may increase in the group size, although this might in many cases be the realistic account of what should happen in the real world. Also, cost-sharing is very common in such contexts—As justified by Weesie and Franzen (1998), the individual's cost of volunteering may well decrease in the number of volunteers showing up, because the collective good can be thus produced in a collaborative manner. However, the cost-sharing structure (a hyperbolic one) proposed by Weesie and Franzen (1998), though providing an incentive for players to cooperate, does not produce a realistic prediction to account for the real-world situations in terms of the comparatives statics of the probabilities as mentioned just now, and thus failed to explain the experiment results provided in the same paper. Since the focus of past studies on the volunteer's dilemma has been centered around the examination of the individual and collective incentives to volunteer under the impact of various parameters, such as the volunteer cost, the group size, etc., it is a meaningful task to incorporate the cost-sharing idea into the volunteer's dilemma model which can also give rise to the correct comparative statics that are compatible with the real-world situation as well as the experiment results. Therefore, the major motivation of this paper is to modify the cost cost structure first discussed by Weesie and Franzen (1998) in a plausible and tractable way, so that the individual's incentive still decay in a larger group, but the collective production probability may decrease, increase, or display unmonotonic pattern with respect to the group size depending on the cost-benefit ratio.

2 Diekmann's original model

2.1 The classic model

The model is based on the volunteer's dilemma game described in Diekmann (1985). We begin with a quick recapitulation of the original game. Consider a n-player game ($n \geq 2$) wherein each player has two strategies, volunteer (V) or stand by (S), and a collective good can be produced if at least one player chooses V.[1] The cost to volunteer is $c > 0$ for all

[1]Theoretically, we can generalize the requirement on the number of volunteers needed for production to, say, that $m \geq 1$ volunteers are needed for production. However, this parameter turns out to affect the order of the equation in solving the mixed-strategy equilibrium and higher-order polynomial will render the equilibrium unsolvable in terms of an explicit (and manageable) solution.

players, who will get a benefit b, $b > c$, if the collective good is produced. There is no cost if the player chooses to stand by, and no benefit if the collective good is not produced (i.e., in the case when no one volunteers). Therefore, when $n = 1$, there is a single player, and he will act optimally by choosing V to get the payoff $b - c$. However, when $n \geq 2$, the player will free ride on other players if someone else has chosen V and in this way gain the full benefit b. One can contextualize the game as n passers-by witnessing a public attack who then need to decide whether to intervene the violence or to stand by in a wish that someone else will step out. The cost of volunteering c, in this context, includes the labor and the possibility of being hurt by the perpetrator during ones act of justice, while the benefit b consists mainly of ones moral relief that the crime is stopped and the victim saved. The basic assumption indicates that the passer-by has such moral standards that oblige him to intervene if he happens to be the only witness.

A standard game theoretic analysis leads quickly to the conclusion that every pure-strategy Nash equilibrium for this basic game involves one player volunteering and all others standing by. Specifically, it comes from the observation that each player volunteers when everyone else stands by, and stands by if anyone else volunteers, so one and only one player volunteering constitutes the unique mutually best-responding Nash equilibrium strategy vector. With n players there exist n such pure-strategy Nash equilibria, raising the question of role assignment (or equilibrium selection) as to who should be the one who volunteers while all other players free ride on him. Therefore, a more pertinent solution concept is the (symmetric) mixed-strategy equilibrium where each player assigns the probability p to choosing V and probability $(1-p)$ to choosing S. By assuming symmetry in the equilibrium, the expected payoffs associated with the two strategies can be written as:

$$u(V \mid p) = b - c, \quad u(S \mid p) = b\left(1 - (1-p)^{n-1}\right)$$

where $(1 - (1-p)^{n-1})$ is the probability that at least one among other $(n-1)$ players has chosen V so that the collective good is produced. Equating two payoffs (i.e., in the equilibrium the player should be indifferent between choosing V or S) gives the mixed-strategy equilibrium $\tilde{p}^*$, and also the probability of the collective good being produced (i.e., the probability that at least one player chooses V) denoted by $\tilde{P}^* = 1 - (1-\tilde{p}^*)^n$:

$$\tilde{p}^* = 1 - \left(\frac{c}{b}\right)^{\frac{1}{n-1}}, \quad \tilde{P}^* = 1 - \left(\frac{c}{b}\right)^{\frac{n}{n-1}}.$$

Hence both the individual's incentive to volunteer $\tilde{p}^*$ and the collective probability of production $\tilde{P}^*$ decrease in the cost-benefit ratio $\frac{c}{b}$ and the group size n. When $n \to \infty$, $\tilde{p}^* \to 0$ and $\tilde{P}^* \to 1 - \frac{c}{b}$. This limit result reveals a shockingly cruel extremity of the free-riding problem, that when the number of witnesses grow sufficiently large, each witness almost certainly chooses to stand by; moreover if the volunteering cost c is close to b, it is almost certain that nobody will step out! The former result, that individual's incentive erodes easily with expanding group size, is to some extent explicable; but the latter result, that the crowd collectively decides not to volunteer when the group size is sufficiently large, is hardly convincing, and seems to be either against reality or against morality. Next, we will first bring up a welfare comparison question and then propose a cost-sharing model to account for this latter claim.

2.2 Welfare comparison for the pure-strategy and mixed-strategy equilibria

As mentioned earlier, if one looks at the pure-strategy (instead of mixed-strategy) Nash equilibrium for this basic game, the unique outcome is for one player to volunteer and for all other players to stand by, because this is the only situation where each player's action forms a best response to all other players' action choices. Despite the role-assignment difficulty, an interesting question to raise is whether the pure-strategy equilibrium delivers higher social welfare than the mixed-strategy equilibrium, as the social welfare of the former is independent of which player volunteers in the equilibrium.

In the pure-strategy equilibrium, the victim is saved and thus every player gets a benefit of b, which is without of generality normalized to $b = 1$. And there is one volunteer who pays the cost c, $0 < c < 1$. Therefore, the social welfare is

$$\tilde{W}_{PS} = n - c.$$

In the mixed strategy equilibrium, the social welfare has the following expression:

$$
\begin{aligned}
\tilde{W}_{MS} &= \sum_{k=1}^{n}(n-kc)\Big(C_n^k p^k (1-p)^{n-k}\Big) \\
&= n - n(1-p)^n - c\sum_{k=1}^{n} k\Big(C_n^k (p)^k (1-p)^{n-k}\Big) \\
&= n - n(1-p)^n - c\sum_{k=1}^{n} np\Big(C_{n-1}^{k-1} (p)^{k-1} (1-p)^{n-k}\Big) \\
&= n - n(1-p)^n - ncp,
\end{aligned}
$$

where $\Big(C_n^k p^k (1-p)^{n-k}\Big)$ is the binomial probability of k players volunteering in the equilibrium, while $(n-kc)$ is the corresponding social welfare. The equilibrium probability of volunteering has been solved earlier:

$$
p = \tilde{p}^* = 1 - (c)^{\frac{1}{n-1}}\ .
$$

Replacing p with $\tilde{p}^*$ in $\tilde{W}_{MS}$, one can get an expression of $\tilde{W}_{MS}$ as a function of n and c, which can be compared with $\tilde{W}_{PS}$. It is expected from the comparison that pure-strategy equilibrium should deliver higher social welfare in its avoidance of duplicate volunteering costs. Indeed, the inequality $\tilde{W}_{PS} > \tilde{W}_{MS}$ always holds for $0 < c < 1$ and $n \geq 2$.[2]

3 Cost-Sharing Volunteer's Dilemma

3.1 The model with a new cost structure

The additional assumption we incorporate into the model is that, instead of each paying c to volunteer, the volunteer's cost decreases when more other players choose V in alliance. Cost-sharing is not a new idea in the study of volunteer's dilemma. Weesie and Franzen (1998) provided several justifying contexts for this cost-sharing assumption: the chance of being hurt by the perpetrator is lower when more passers-by come to intervene; one

[2]The inequality is justified by graphical simulation instead of direct analytical comparison due to intractability.

volunteer needs to exercise less labor for the maintenance of a community center when more volunteers show up; the embarrassment of complaining about someone's violation of public rule (e.g., talking in quiet-study areas) is lessened if other people join in the complaints... In general, cost-sharing alleviates one's burden to comply with social morality in large group and encourages cooperation. Indeed, the two cost structures proposed in Weesie and Franzen (1998), namely evenly splitting the cost among volunteers as one and randomly drawing one volunteer to pay the full cost as another, both saw a theoretical increase in the probability of (at least) one player volunteering compared to a game where the cost is not shared, but the comparative statics, i.e., how the probabilities change with the group size, remain the same as Diekmann's model. And the theory prediction is inconsistent with their experiment results given in the same paper: despite the limited variation of n in the experiment ($n = 2, 4, 8$), "the observed macro probabilities of public goods production are almost always larger in the 8-person groups than in the smaller groups."

Is the theoretical prediction a universal one for any possible cost structure? It turns out not to be so; in fact, the property of the comparative statics seems to depend on *how the cost is shared* among players. Next, we propose a new cost structure which generates nicer equilibrium comparative statics that may account for the inconsistency exhibited in the "macro probabilities of public goods production." For presentation simplicity, we normalize the benefit $b = 1$, and let the cost satisfy $0 < c < 1$. When there are k volunteers showing up, Weesie and Franzen (1998) assume that they split the cost c so that each volunteer incurs cost $\frac{c}{k}$; alternatively, we assume each volunteer incurs cost c^k. Both are simplifying, certainly plausible, and also tractable specifications of the cost structure.[3]

Assumption. *If there are k volunteers, $n \geq k \geq 1$, then the cost to each volunteer is c^k for choosing V, where $1 > c > 0$.*

Notice $c^k = \frac{c}{k}$ when $k = 1$, and $c^k < \frac{c}{k}$ when k grows sufficiently large (almost immediately for small c's, such as for $c < 0.5$). Hence our assumption implies extra efficiency in production associated with cooperation, e.g., the individual risk of getting hurt drops by more than two-thirds if three passers-by unitedly intervene a public violence, one volunteer needs to

[3]Indeed, due to the mathematical property of the game, we cannot find another tractable way to model cost-sharing in this context except the hyperbolic function given in Weesie and Franzen (1998) and the exponential function studied here.

exercise less than half effort in community maintenance as before if he collaborates with another volunteer, etc. Thus it is a stronger form of cost sharing. Imposing symmetry in equilibrium, then the expected payoffs associated with the two strategies are:

$$u(V \mid p) = 1 - \sum_{k=0}^{n-1} c^{k+1} C_{n-1}^{k} p^{k} (1-p)^{n-1-k}, \quad u(S \mid p) = 1 - (1-p)^{n-1}$$

where p is the (symmetric) probability of choosing V for each player, index k of the summation is the number of other volunteers out of $(n-1)$ players except the focal player, C_{n-1}^{k} is the combinatorial number for selecting k players out of $(n-1)$ players. Notice that $u(V \mid p) = 1 - c\sum_{k=0}^{n-1} C_{n-1}^{k}(cp)^{k}(1-p)^{n-1-k} = 1 - c(cp+1-p)^{n-1}$, the latter equality coming from the binomial theorem formula. By equating $u(V \mid p)$ and $u(S \mid p)$, we can solve for the equilibrium p^*, and then derive the probability of the collective good being produced $P^* = 1 - (1-p^*)^n$ accordingly. Simple algebra gives rise to the next Theorem.

Theorem 1. *Under the exponential cost structure, there is a unique mixed-strategy equilibrium where the individual probability of volunteering p^* and the probability of the collective good being produced P^* are:*

$$p^* = 1 - \frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1}, \quad P^* = 1 - \left(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1}\right)^{n}. \tag{1}$$

The proof to Theorem 1 follows the preceding paragraph and details are omitted. Notice that the model's tractability relies on the use of the binomial theorem formula and thus only applies to the hyperbolic/exponential function form; moreover, if we change the assumption to require more than one volunteer's participation for the good's production (e.g., the violence can only be stopped when two or more passers-by intervene), then the index k of the summation in $u(V \mid p)$ should starts from a number greater than 0, and then the equation $u(V \mid p) = u(S \mid p)$ becomes a higher order polynomial of p which is unsolvable. Therefore, although there are other possible ways to model the cost structure, tractability confines us to this specific exponential form, which is, nonetheless, sufficient to generate a more general comparative statics pattern.

Theorem 2. *The equilibrium probabilities have the following properties.*

1. *Both p^* and P^* decreases in c.*
2. *p^* decreases in n.*
3. *There exist $0 < c_1, c_2 < 1$ such that:*

   i) when $0 < c < c_1$, P^* increases in n for all $n \geq 3$.[4]

   ii) when $c_1 < c < c_2$, P^* decreases in n first and then increase in n.

   iii) when $c_2 < c < 1$, P^* decreases in n for all $n \geq 3$.

Compared to Weesie and Franzen (1998), which arrived at the conclusion that P^* is globally decreasing in n for all parameter ranges, Theorem 2 allows P^* to increase partially or almost globally in n for sufficiently small c, although when c is sufficiently large (as in 3.(iii)) the two models' conclusion coincide. Therefore, our results provide a potential justification for the abnormality found in Weesie and Franzen's (1998) experiment data. Put in a realistic context, the Theorem suggests that given the exponential cost structure, whether a larger crowd favors the production of the collective good depends on the magnitude of the cost parameter of volunteering (i.e., c). That is, if the perpetrator appears to be weak and easily restrained, the violence is more likely to be stopped when there are more witnesses. If the work load of the community center's maintenance is relatively light, the center is more likely to be well maintained with a larger community size. The opposite holds if the perpetrator appears ferocious or the workload seems daunting. Except the absolute magnitude of the cost, the comparison of our model and Weesie and Franzen's (1998) model suggests that the individual and collective incentives for volunteering are also affected by the way the costs are shared among all volunteers. As mentioned earlier, the exponential cost structure is stronger than the evenly-splitting cost structure, and consequently, such extra efficiency in cooperation makes the collective good more likely to be produced in a larger group.

3.2 Welfare comparison with cost sharing

In the classic version of the volunteer's dilemma, we have conducted welfare comparison for the pure-strategy and mixed-strategy equilibria, and confirmed that $\tilde{W}_{PS} > \tilde{W}_{MS}$ always

[4]Notice that we impose $n \geq 3$ instead of $n \geq 2$. In fact, P^* always decreases in n at the point $n = 2$, but may start to increase when n is slightly larger than 2 so that discretely P^* is greater at $n = 3$ than at $n = 2$. To avoid mathematical complication we ignore P^*'s property at the single point $n = 2$.

holds as the former avoids duplicate volunteering costs by having a definite number of volunteers, i.e., one, in every equilibrium outcome. The same comparison question can be asked here, with cost sharing, but now one needs a second thought on the answer because of the advantage of cost savings when two or more players volunteer.

Despite the cost-sharing efficiency, the pure-strategy social welfare should be the same as in the classic model, because only one player volunteers in any equilibrium. Thus

$$W_{PS} = \tilde{W}_{PS} = n - c.$$

For the mixed strategy equilibrium, we have

$$\begin{aligned}
W_{MS} &= \sum_{k=1}^{n} (n - kc^k)\Big(C_n^k p^k (1-p)^{n-k}\Big) \\
&= n - n(1-p)^n - \sum_{k=1}^{n} k\Big(C_n^k (cp)^k (1-p)^{n-k}\Big) \\
&= n - n(1-p)^n - \sum_{k=1}^{n} ncp\Big(C_{n-1}^{k-1} (cp)^{k-1} (1-p)^{n-k}\Big) \\
&= n - n(1-p)^n - ncp(1 + cp - p).
\end{aligned}$$

where $\Big(C_n^k p^k (1-p)^{n-k}\Big)$ is again the binomial probability of k players volunteering in the equilibrium, while $(n - kc^k)$ is the corresponding social welfare with cost-sharing (reflected in c^k as a decreasing function of k). And the equilibrium probability has been solved to be

$$p = p^* = 1 - \frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1}.$$

Due to W_{MS}'s complex expression, we resort to graphical simulation to graph W_{MS} and W_{PS} as functions of n and c for a magnitude comparison. The results show that $W_{PS} > W_{MS}$ in most case, but $W_{PS} < W_{MS}$ for sufficiently small c (approximately for $c < 0.38$) with small n as well, when the cost-sharing effect is expected to be the strongest.

4 Conclusion

We examined the volunteer's dilemma first proposed in Diekmann (1985) and extensively studied in several following papers written by Weesie and his coauthor. The motivation of our paper is to find a modification of the cost structure discussed in Weesie and Franzen (1998) so that the game's equilibrium can account for the possibility that a larger group size may favor the production of the collective good, which is a plausible scenario in some practical contexts and also partially explains the abnormality of the experiment data given in the same paper. Our result suggests that the characteristic of the individual and the collective incentives to volunteer are not only affected by the absolute magnitude of the volunteering cost but also by the way the cost is shared among all volunteers, i.e., how efficiently the volunteers collaborate with each other. Due to the mathematical property of the game, it is hard to further generalize the cost structure without compromising the analytical tractability.

References

[1] Diekmann, A. (1985). Volunteer's dilemma. *Journal of conflict resolution*, 29(4), 605-610.

[2] Weesie, J. (1993). Asymmetry and timing in the volunteer's dilemma. *Journal of conflict resolution*, 37(3), 569-590.

[3] Weesie, J. (1994). Incomplete information and timing in the volunteer's dilemma: A comparison of four models. *Journal of conflict resolution*, 38(3), 557-585.

[4] Weesie, J., & Franzen, A. (1998). Cost sharing in a volunteer's dilemma. *Journal of conflict resolution*, 42(5), 600-618.

5 Appendix

Proof of Theorem 2

First, let us show that both p^* and P^* decrease in c. Since $P^* = 1-(1-p^*)^n$, it suffices to show that p^* decreases in c, which follows immediately from the transformation:

$$p^* = 1 - \frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1} = 1 - \frac{1}{\frac{1}{c}\left(\left(\frac{1}{c}\right)^{\frac{n}{n-1}} - 1\right) + 1}.$$

And that p^* is a decreasing function of n (given that $c < 1$) is also straightforward.

Now, we prove the comparative statics of P^* with respect to n. Differentiate P^* with respect to n in Eq. (1):

$$\frac{dP^*}{dn} = -\left(\frac{1}{\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c}}\right)^n \left(\frac{n\left(\frac{1}{c}\right)^{\frac{n}{n-1}} \log(\frac{1}{c})}{(n-1)^2\left(\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c}\right)} + \log\left(\frac{1}{\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c}}\right)\right).$$

Notice $\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c} = \frac{1}{c}\left(\left(\frac{1}{c}\right)^{\frac{1}{n-1}} - 1\right) + 1 > 1$ because $\frac{1}{c} > 1$, thus $\left(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1}\right)^n$ is positive. For the second bracketed term, let us define

$$f(n) = \frac{n\left(\frac{1}{c}\right)^{\frac{n}{n-1}} \log(\frac{1}{c})}{(n-1)^2\left(\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c}\right)} \text{ and } g(n) = -\log\left(\frac{1}{\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c}}\right), \tag{2}$$

so the second bracketed term equals $f(n) - g(n)$. Now it is clear that $f(n) > 0$ and $g(n) > 0$, and the sign of $\frac{dP^*}{dn}$ depends on the sign of $-\Big(f(n) - g(n)\Big)$ or $g(n) - f(n)$. To sign $\frac{dP^*}{dn}$, we need to study the properties of $g(n)$ and $f(n)$. It is straightforward to see that $g(n)$ decreases in n, but other than this, it is hard to sign $f'(n)$, $f''(n)$ and $g''(n)$ analytically. However, since the only parameter is c which ranges between 0 and 1, using simulation, we confirm that both $g(n)$ and $f(n)$ are decreasing convex functions of n for all $0 < c < 1$.

We first show that there exists some c_1, supposedly to be a small number close to 0, such that when $0 < c < c_1$, $f(n) < g(n)$, so that P^* increases in n for all $n \geq 3$. It is hard to directly solve the inequality and find c_1, but we can inspect the values of $f(n)$ and $g(n)$ when $c \to 0$. Indeed, we can show that $f(n) < g(n)$ for all $n \geq 3$ when $c \to 0$ by the following arguments. Notice when $c \to 0$, $g(n) \to -\log(\frac{0}{\infty - 1}) = -\log(0) = \infty$, while

$$f(n)=\frac{n}{(n-1)^2}\frac{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\log(\frac{1}{c})\to\infty.$$

Thus we can use L'Hospital's Rule to prove $\frac{f(n)}{g(n)}<1$ when $c\to 0$. Transforming the inequality, $\frac{f(n)}{g(n)}<1$ is equivalent to

$$\frac{n}{(n-1)^2}\left(\frac{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right)\left(\frac{\log(\frac{1}{c})}{-\log(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1})}\right)<1.$$

When $c\to 0$, $\left(\frac{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right)\to 1$, while both $\log(\frac{1}{c})\to\infty$ and $-\log(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1})\to\infty$. Use L'Hospital's Rule twice for the second bracketed term and one gets:

$$\frac{\log(\frac{1}{c})}{-\log(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1})}\to\frac{n-1-(c)^{\frac{n}{n-1}}(n-1)^2}{n}\to\frac{n-1}{n}.$$

Thus

$$\frac{f(n)}{g(n)}\to\frac{n}{(n-1)^2}\frac{n-1}{n}=\frac{1}{n-1}<1,\quad \text{when } c\to 0,\ \text{for all } n\geq 3.$$

Now that $f(n)<g(n)$ holds strictly for all $n\geq 3$ when $c\to 0$, and also $f(n)$ and $g(n)$ are continuous in the parameter c for all $0<c<1$, there must exist some $c_1>0$, supposedly small, such that $f(n)<g(n)$ for all $0<c<c_1$. The proof is now complete for this case.

Next, we show that there exists some $c_2>c_1$ such that for $c_1<c<c_2$, there exists some $\tilde{n}$ such that $f(n)>g(n)$ for $n<\tilde{n}$ and $f(n)<g(n)$ for $n>\tilde{n}$, so P^* decreases in n first and increases in n thereafter. We need three steps. First, we can show $f(2)>g(2)$ to be always true. Second, we can show $f(n),g(n)\to 0$ when $n\to\infty$. Third, we want to show that $f(n)$ and $g(n)$ cross exactly once when n increases from 2 to infinity. To show the third step, we can do the following. We can show $f'(2)<g'(2)$ and $f'(\infty)>g'(\infty)$ when c is sufficiently small. Then since $f(n)$ and $g(n)$ are smooth convex functions, their derivatives $f'(n)$ and $g'(n)$ must cross at least once when n increases from 2 to infinity. Suppose they last cross at n^*, which implies $0>f'(n)>g'(n)$ for all $n>n^*$. Now that $f(\infty)=g(\infty)=0$, and g decreases faster than f for all $n>n^*$, it must be that $f(n)<g(n)$ for all $n\geq n^*$. Now that

we have $f(2) > g(2)$, $f(n) < g(n)$ for all $n \geq n^*$, and $f(\infty) = g(\infty) = 0$, combined with the fact that f and g are smooth convex functions, we can conclude that f and g cross only once at some $\tilde{n}$ smaller than n^*. And the proof should be complete for this case. Therefore, we need to prove the following: (i) $f(2) > g(2)$; (ii) $f(\infty) = g(\infty) = 0$; (iii) $f'(2) < g'(2)$ and $f'(\infty) > g'(\infty)$ when c is sufficiently small. Now let us do it step by step.

(i) $f(2) > g(2)$, i.e., we want to show $\frac{2\left(\frac{1}{c}\right)^2 \log(\frac{1}{c})}{\left(\left(\frac{1}{c}\right)^2+1-\frac{1}{c}\right)} > -\log(\frac{1}{\left(\frac{1}{c}\right)^2+1-\frac{1}{c}})$. Let $a = \frac{1}{c} > 1$. We want to show $\frac{2a^2\log(a)}{(a^2+1-a)} > -\log(\frac{1}{a^2+1-a}) = \log(a^2+1-a)$, that is, $2a^2\log(a) - (a^2+1-a)\log(a^2+1-a) > 0$, that is, $a^2\log(a^2) - (a^2+1-a)\log(a^2+1-a) > 0$. Notice the function $x\log(x)$ is increasing in x. Since $a = \frac{1}{c} > 1$, so $a^2 > a^2+1-a$, and thus $a^2\log(a^2) > (a^2+1-a)\log(a^2+1-a)$, and the inequality is proved.

(ii) $f(\infty) = g(\infty) = 0$. When $n \to \infty$, since $\left(\frac{1}{c}\right)^{\frac{1}{n-1}} \to \left(\frac{1}{c}\right)^0 = 1$,

$$f(n) = \frac{n\left(\frac{1}{c}\right)^{\frac{1}{n-1}}\log(\frac{1}{c})}{(n-1)^2\left(\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1\right)} \to 0,$$

$$g(n) = -\log(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}) \to -\log(\frac{c}{c}) = 0.$$

(iii) $f'(2) < g'(2)$ and $f'(\infty) > g'(\infty)$ when c is sufficiently small. Since $f' < 0$ and $g' < 0$, we want to show $\frac{f'(2)}{g'(2)} > 1$ and $\frac{f'(\infty)}{g'(\infty)} < 1$.

$$\frac{f'(n)}{g'(n)} = \frac{n^2-1+\frac{(c-1)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}n}{(n-1)^2}.$$

So

$$\frac{f'(n)}{g'(n)} >/< 1 \iff n^2-1+\frac{(c-1)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}n >/< (n-1)^2 \iff$$

$$2n-2 >/< \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}n \iff n\left(2-\frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right) >/< 2.$$

When $n=2$, we can show $\frac{(1-c)\log(\frac{1}{c})}{\frac{1}{c}+c-1} < 1$ for all c, so $2\left(2-\frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)+c-1}\right) > 2$ and thus $\frac{f'(2)}{g'(2)} > 1$ for all c. The proof goes as following. Let $b = \frac{1}{c} > 1$, then $\frac{(1-c)\log(\frac{1}{c})}{\frac{1}{c}+c-1} = \frac{\left(1-\frac{1}{b}\right)\log(b)}{b+\frac{1}{b}-1}$.

We want to show $\frac{\left(1-\frac{1}{b}\right)\log(b)}{b+\frac{1}{b}-1} < 1 \iff \left(1-\frac{1}{b}\right)\log(b) < b+\frac{1}{b}-1 \iff (b-1)\log(b) < b^2-b+1 = b(b-1)+1$. Notice $\log(b) < b$ for all $b > 1$, thus $(b-1)\log(b) < b(b-1) < b(b-1)+1$, and $\frac{(1-c)\log(\frac{1}{c})}{\frac{1}{c}+c-1} < 1$ is proved. Thus we have shown $f'(2) < g'(2)$ for all c.

Now examine the case of $n \to \infty$. Recall $\frac{f'(n)}{g'(n)} < 1$ if $\left(2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right) n < 2$, and when $n \to \infty$, the LHS goes to either ∞ or $-\infty$ depending on the sign of the constant $\left(2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right)$. Since $\left(\frac{1}{c}\right)^{\frac{1}{\infty}} \to 1$, so $2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1} \to 2 - \frac{(1-c)\log(\frac{1}{c})}{c}$, and thus we need to examine the sign of this expression with respect to c. Notice that the derivative of $\frac{(1-c)\log(\frac{1}{c})}{c}$ with respect to c is $\frac{-1+c-\log(\frac{1}{c})}{c^2} < 0$, so it decreases in c. When $c \to 0$, $\frac{(1-c)\log(\frac{1}{c})}{c} \to \frac{\log(\infty)}{0} = \infty$. When $c \to 1$,$\frac{(1-c)\log(\frac{1}{c})}{c} = 0$. Due to monotonicity, there exists some $c_2 > 0$ such that $\frac{(1-c_2)\log(\frac{1}{c_2})}{c_2} = 2$. And for all $c < c_2$, $\frac{(1-c)\log(\frac{1}{c})}{c} > 2$, so $\lim_{n\to\infty}\left(2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right) < 0$ and thus $\left(2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right) n \to -\infty$, so $\frac{f'(\infty)}{g'(\infty)} < 1$, or $f'(\infty) > g'(\infty)$. Vice versa, for all $c > c_2$, $\frac{f'(\infty)}{g'(\infty)} > 1$, or $f'(\infty) < g'(\infty)$.

Now that for $c < c_2$, we have shown (i)(ii)(iii) for $c < c_2$. Still, we need to verify that $c_1 < c_2$. Recall that c_1 is a number any close to zero such that when $0 < c < c_1$, we have $f(n) < g(n)$ for all $n \geq 3$ given that (1) $\lim_{c\to 0} f(n;c) < \lim_{c\to 0} g(n;c)$ for all $n \geq 3$ and (2) $f(n;c), g(n;c)$ are continuous in c. Therefore, we only need to verify that $f(n) > g(n)$ for all $n \geq 3$ at c_2, and then by definition $c_1 < c_2$. Recall c_2 is defined implicitly by $\frac{(1-c_2)\log(\frac{1}{c_2})}{c_2} = 2$, which can be solved by numerical approximation to be $c_2 \approx 0.3464$, and by simulation one can easily see that $f(n) > g(n)$ for all $n \geq 3$ at $c_2 \approx 0.3464$, so it follows that $c_1 < c_2$. Lastly, note that the proof stated above leads to the conclusion that P^* decreases in n first and increases in n thereafter for all $c < c_2$, including those $c < c_1$, but this does not conflict with the first case. The reasoning we followed so far points to the intuition that for P^* to exhibit an increasing interval with respect to n, a smaller c is more favorable. In fact, when $c < c_1 < c_2$, it can be seen by simulation that the threshold dividing the decreasing and increasing parts of $P^*(n)$ is less than 3, meaning P^* only decreases before $n = 3$ and increases for all $n \geq 3$, so there is no conflict between 3.(i) and 3.(ii) of Theorem 2.

Lastly, we need to prove the third case, that P^* decreases for all $n \geq 3$ when $c > c_2$. In the second case, we have already proved that $f(2) > g(2)$, $f(\infty) = g(\infty) = 0$, $f'(2) < g'(2)$

for all c, and $f'(\infty) < g'(\infty)$ when $c > c_2$ (the other case). For smooth convex functions $f(n)$ and $g(n)$ it is only possible that $f(n) > g(n)$ for all $n \geq 2$ (which is confirmed by simulation), though in Theorem 2 we relaxed it to $n \geq 3$ for consistency. So 3.(iii) is proved, and the proof is complete. Q.E.D