Volunteer's Dilemma: Cost-Sharing Revisited

Rabah Amir[*], Dominika Machowska[†], Jingwen Tian[‡]

March 18, 2024

Abstract

This paper revisits the well-known volunteer's dilemma on the production of a collective good when a single participant is sufficient for the task. We propose a cost-sharing model with a volunteering cost that decreases exponentially in the number of volunteers. We show that, at the unique mixed-strategy equilibrium, the probability of production may increase in the number of players for sufficiently low volunteering costs. This provides an alternative account of the fit of the model with some real-life situations and a better fit with the associated experimental results, that is, a larger group does erode the individual incentive to volunteer but in an offsetting way that favors the production of the collective good. A second result is that the mixed-strategy equilibrium may be more socially efficient than the pure-strategy equilibrium for some parameter values, which is a major reversal with respect to the standard dilemma and many other coordination games.

Keywords: bystander effect, coordination games, free-riding, mixed strategy

[*]Department of Economics, the University of Iowa, Iowa City, IA, USA. Email: rabah-amir@uiowa.edu.

[†]Institute of Applied Mathematics and Mechanics University of Warsaw, Poland. Email: machowska@mimuw.edu.pl.

[‡]Department of Economics, the University of Iowa, Iowa City, IA, USA. Email: gin-tian@uiowa.edu.

1 Introduction

The volunteer's dilemma (Diekmann, 1985) is a n-player, complete-information, simultaneous-move game dealing with the production of a collective good or task that requires a single player or volunteer to produce. Each individual makes a binary decision to volunteer (at a cost c) or not, knowing that a benefit b accrues to each player if the good is produced. The latter outcome requires only (at least) one player to volunteer, with more players adding cost without accruing any extra benefit to anyone. In general, $b > c$, so an individual would have an incentive to volunteer if she were the only player in the game. However, when other players are present, this individual incentive to volunteer might be eroded by the hope that someone else will step in to accomplish the task so that she can obtain the same benefit at no cost. in other words, the game provides an opportunity for free-riding. This incentive to free-ride produces equilibrium outcomes that are socially inefficient.

The volunteer's dilemma game or closely related variants often emerge in a diversity of political, economic or military settings, including for instance Bliss and Nalebuff (1984), Konrad and Morath (2021), and Konrad (2024).[1] In particular, the volunteer's dilemma game is often associated with a real-life story in 1960s New York City of a lady named Kitty Genovese, who was assaulted and murdered by her building with 38 witnesses who either saw or heard the attack without any of them calling the police, until it was too late.[2] Subsequent coverage by *The New York Times* called public attention to the problem of the *bystander effect,* well known in social psychology. However, as Weesie (1993) surmised, it may not have been the immorality or indifference of the witnesses that killed Ms. Genovese, but rather their rational free-riding behavior, as all witnesses likely assumed someone else would call the police. Then a natural question is, would Kitty Genovese have been more fortunate if there had been only 5 witnesses, or 500 witnesses? In other words, would a larger group favor or disfavor the production of the collective good, when there is reduced individual incentive but increased chance of *someone* stepping in?

It is a priori hard to imagine that she is less likely to be saved with 500 witnesses than with 38 witnesses, which only happens if every one of them abstains from taking any action; How likely is that to take place? Moreover, experimental results reported in Weesie and

Franzen (1998) do not provide any grounding for the conjecture that the success probability (that the collective good is produced) should decrease in the number of players; in fact, some of the results point to the opposite (see Table 4 in Weesie and Franzen, 1998). However, Diekmann's (1985) original model, though artfully revealing the rationale and inefficiency behind the volunteer's dilemma, predicts that the production probability does decrease in the number of players, unless people assume, in line with Kant's categorical imperative, that everyone's actions are bound by the exact same morality (a universal law), so that when one player chooses an action, he assumes all other players will make the same choice. This philosophical notion is in some sense too strong, goes against the modern economic rationality principle and thus against the whole spirit of game theory.

Following Diekmann (1985), other authors have extended the original analysis in several directions. Weesie (1993) proposed an endogenous *timing* scheme to Diekmann's static game, where players make the decision of *when* to step in, given other players have not done so yet, and showed that the asymmetric version (in terms of players' cost-to-gain ratio) of this timing game solves the dilemma in the sense that the most favorable player will volunteer immediately. Weesie (1994) further shows that the observability of the behavior of other players (allowed in the timing game) and the uncertainty with respect to the payoffs of other players both enhance the likelihood that a player volunteers and the production probability, except when uncertainty is high. In such a game, a larger group may favor the good's production for *sufficiently* large n. Lastly, Weesie and Franzen (1998) brought in the novel idea of cost-sharing, by making the cost to volunteer divisible among all the volunteers either by even splitting (i.e., each volunteer incurs cost $\frac{c}{k}$ when k players volunteer) or by a lottery to select one volunteer to carry the full cost burden. They concluded using theoretical analysis that each individual is less likely to volunteer and that the collective good is less likely to be provided, in larger groups. However, their experimental results only partially support the former conclusion while largely refuting the latter. Finally, a full-fledged laboratory (experimental) test of the standard volunteer's dilemma confirmed that volunteer rates are lower with larger groups, but also found that the incidence of no-volunteer outcomes declines with group size, in contrast to the volunteer's dilemma theory (see Goeree et al., 2017). These findings are also reinforced in the main by other experimental studies of the bystander effect, e.g., by Campos-Mercade (2021).

Therefore, except by resorting to the intricate philosophical notion of Kant's imperative or by bringing in incomplete information (and then only for sufficiently large n), to our knowledge, there is yet no compelling model in the related literature to account for the possibility that the production probability of the collective good may increase with group size. Yet, it is fair to say that the latter conclusion might well be in many cases the more realistic outcome in relevant real-world situations. As justified by Weesie and Franzen (1998), an individual's cost of volunteering may well decrease in the number of volunteers taking action, because the collective good can then be produced at lower individual cost in a collaborative manner. For instance, in random attacks of the type Kitty Genesove was subjected to, and taking place in public, it will often be much less costly for a group of volunteers to overcome the assailant than for a single person to do so. In the drowning kid example, a rescue by a group would also be typically easier and safer to carry out than by a single person. Indeed, the idea of cost-sharing is easily seen to be natural in a variety of relevant contexts for the volunteer's dilemma.[3]

However, the equal cost-sharing structure (a hyperbolic one) proposed by Weesie and Franzen (1998), though providing an incentive for players to cooperate, does not produce a realistic prediction to account for many of the relevant real-world situations in terms of the comparative statics of the volunteering probabilities as just mentioned. In addition, this cost structure fails to explain the experiment results provided in the same paper. Since the focus of past studies on the volunteer's dilemma has been centered around the examination of the individual and collective incentives to volunteer under the impact of various parameters, such as the volunteering cost, the group size, etc., it is a meaningful task to further explore the cost-sharing idea for the volunteer's dilemma model, which can also give rise to the correct comparative statics that are compatible with relevant real-world situations as well as with experiment results.

Therefore, the major motivation of this paper is to modify the cost structure first discussed by Weesie and Franzen (1998) in a plausible and tractable way, so that the individual incentive to act still declines with group size, but the collective production probability may decrease, increase, or display a non-monotonic pattern with respect to the group size depending on the cost-benefit ratio. To do so, we propose an individual cost function for volunteering that decays exponentially in the number of participants. We argue below that this cost structure is a plausible one for a variety of settings to which the game at hand a priori applies. Furthermore, the cost structure is also analytically tractable, which is of course a natural requirement, since the attending computational complexity for mixed-strategy equilibrium and its comparative statics properties could easily turn out to be prohibitive. The main result of this paper establishes that this choice of cost function essentially delivers our desired conclusions on the collective production probability for the public good. These in turn are now consistent with the findings from the experimental studies that are cited above.

The rest of this paper is organized as follows. Section 2 reviews the standard model and provides a summary of its results, along with a novel formal welfare analysis and comparison. Section 3 lays out our modification to the basic model and its associated analysis, including a similar welfare comparison. Section 4 is a brief conclusion.

2 Diekmann's original model

This section provides an overview of the classical model of Diekmann (1985) and of its main results. This is followed by an analysis of the welfare properties of the model, along utilitarian lines. While elementary, this welfare analysis is a novel contribution to the related literature, to the best of our knowledge.

2.1 The classical model

This subsection provides a brief but self-contained review of the basic model of the volunteer's dilemma. The original model is the volunteer's dilemma game introduced by Diekmann (1985). We begin with a quick recapitulation of the original game. Consider a n-player game ($n \geq 2$) wherein each player has two strategies, volunteer (V) or stand by (S), and a collective good can be produced if and only at least one player chooses V.[4] The cost to volunteer is $c > 0$ for each player. Every player will obtain the same benefit b, with $b > c$, if the collective good is produced. There is no cost to any player who chooses to stand by, and no benefit to any player if the collective good is not produced (i.e., in the case when no one volunteers). The game is thus a classical symmetric anti-coordination game[5] with binary actions.

The most common contextualization of this game refers to n passers-by witnessing a person in some distress or danger, who must decide whether to intervene to save the person or to stand by in hope that someone else will step in instead. The cost of volunteering c, in this context, includes the cost of effort, the tacit opportunity cost and the possibility of being physically hurt in the process by the perpetrator during one's intervention. The benefit b consists mainly of one's moral relief that the crime or harm has been prevented and the victim saved. The basic assumption reflects the fact that each passer-by has such moral standards that prompt him to intervene if he happens to be the only witness.

For the special case of a single player (i.e., $n = 1$), she would act optimally by volunteering (or choosing action V) and thus get the payoff $b - c > 0$. However, when $n \geq 2$, a player has the option to free ride on other players' efforts as she might expect someone else to choose V, thus gaining the full benefit b at no cost.

With the general n-player case being a typical game of coordination, a standard simple game-theoretic analysis leads to the conclusion that every pure-strategy Nash equilibrium for this basic game involves one player volunteering and all others standing by. Specifically, this follows from the observation that each player will volunteer when everyone else stands by, and will stand by if at least one other person volunteers. Hence, one and only one player volunteering constitutes the unique Nash equilibrium strategy vector (up to all possible permutations of the players). With n players in the game, there exist n such pure-strategy Nash equilibria, raising the question of role assignment (or equilibrium selection) as to who should be the one to volunteer while all other players free-ride on her.

Alternatively, due to the fully symmetric nature of the game, a more focal solution concept is the (symmetric) mixed-strategy equilibrium where each player assigns probability p to action V and probability $(1 - p)$ to action S. By assuming symmetry in equilibrium, the expected payoffs associated with the two pure strategies can be written as

$$u(V \mid p) = b - c \text{ and } u(S \mid p) = b\left(1 - (1 - p)^{n-1}\right),$$

where $(1-(1-p)^{n-1})$ is the probability that at least one among the other $(n-1)$ players chooses V so that the collective good is produced. Invoking the indifference criterion (i.e., in a mixed-strategy equilibrium, a player should be indifferent between choosing V or S) and equating the two payoffs gives the mixed-strategy equilibrium p^*, as well as the probability of the collective good being produced (i.e., the probability that at least one player chooses V) denoted by $P^* = 1-(1-p^*)^n$:

$$p^* = 1 - \left(\frac{c}{b}\right)^{\frac{1}{n-1}} \text{ and } P^* = 1 - \left(\frac{c}{b}\right)^{\frac{n}{n-1}}.$$

It may be seen by inspection that both the individual's incentive to volunteer p^* and the collective probability of success P^* decrease in the cost-benefit ratio $\frac{c}{b}$ and in the group size n. It is also easy to verify that as $n \to \infty$, $p^* \to 0$ and $P^* \to 1 - \frac{c}{b}$. This limit result reveals two a priori counter-intuitive implications of the free-riding problem: as the number of witnesses grows sufficiently large, each witness will almost certainly choose to stand by; and moreover if the volunteering cost c is close to the benefit b, it is almost certain that nobody will step in! The former result, that individual incentive erodes monotonically with expanding group size, is to some extent not so surprising as a consequence of the free-riding opportunity afforded by the game. The latter result, that the crowd collectively decides not to volunteer when the group size is sufficiently large, is more of an extreme outcome.

We now complete the existing analysis of the original game by adding a novel welfare analysis, which is presented in the next subsection.

2.2 Welfare comparison for the pure-strategy and mixed-strategy equilibria

This subsection adds a welfare comparison of the two different Nash equilibrium points (in pure and mixed strategies) of the volunteer's dilemma. As mentioned earlier, if one looks at the pure-strategy (instead of mixed-strategy) Nash equilibrium for this basic game, the unique outcome (up to player permutations) is for one player to volunteer and for all other players to stand by. Indeed, it is easy to verify that this is the only situation where each player's action forms a best response to all other players' action choices. An interesting question to address is whether the pure-strategy equilibrium delivers higher utilitarian social welfare than the mixed-strategy equilibrium, or vice versa.

In the pure-strategy equilibrium, the victim is saved with certainty and thus every player gets a benefit of b, which is henceforth (without loss of generality) normalized to $b = 1$. In addition, there is exactly one volunteer who pays the cost c, $0 < c < 1$. Therefore, the corresponding utilitarian social welfare, the sum of all players' payoffs, is

$$\tilde{W}_{PS} = n - c.$$

It is important to observe that the pure-strategy equilibrium is Pareto-optimal. In fact, as is easily seen, it corresponds exactly to the first-best solution one would obtain by a social planner aiming to maximize a utilitarian social welfare composed of the sum of the n agents' utilities by picking the number of volunteers to participate.

Under the mixed strategy equilibrium, the expected utilitarian social welfare has the following expression:

$$\begin{aligned}\tilde{W}_{MS} &= \sum_{k=1}^{n}(n-kc)\Big(C_n^k p^k (1-p)^{n-k}\Big)\\ &= n - n(1-p)^n - c\sum_{k=1}^{n} k\Big(C_n^k (p)^k (1-p)^{n-k}\Big)\\ &= n - n(1-p)^n - c\sum_{k=1}^{n} np\Big(C_{n-1}^{k-1}(p)^{k-1}(1-p)^{n-k}\Big)\\ &= n - n(1-p)^n - ncp,\end{aligned}$$

where $\Big(C_n^k p^k (1-p)^{n-k}\Big)$ is the binomial probability of k players volunteering at equilibrium, and $(n-kc)$ is the corresponding social welfare. The equilibrium probability of volunteering has been solved earlier as

$$p = p^* = 1 - (c)^{\frac{1}{n-1}}.$$

The main result here is the following simple observation.

Theorem 1. *The pure-strategy equilibrium delivers higher social welfare than the mixed-strategy equilibrium.*

The proof of this Theorem follows directly by taking the difference between the social welfare of the pure-strategy and the mixed-strategy equilibria. Indeed, we observe that $\tilde{W}_{PS} - \tilde{W}_{MS} = -c + n(1-p)^n + ncp$ and replacing p with p^* in $\tilde{W}_{MS}$, one can get that

$$\tilde{W}_{PS} - \tilde{W}_{MS} = c(n-1) > 0 \text{ for } c \in (0,1) \text{ and } n \geq 2.$$

This conclusion is to be expected on intuitive grounds since pure-strategy equilibrium ensures that the task is accomplished with probability one and does so while avoiding any duplication in volunteering costs. Neither of these features are satisfied by the symmetric mixed-strategy equilibrium. In particular, every number of volunteers from 1 to n arises with positive probability, thus giving rise to a corresponding level of duplication of the volunteering costs (except for the case when $n = 1$), including the maximal level of nc when everyone volunteers. These duplication costs are a pure loss for the group associated with mixed strategy and thus give rise to social inefficiency.

In the next section, we propose a cost-sharing model that is more realistic for many of the possible applications of this model, as noted earlier. This new cost structure also partly mitigates the social inefficiency due to cost duplication.

3 Volunteer's dilemma with cost-sharing

This section proposes a new cost function for the volunteer's dilemma while keeping the rest of the game the same. The new feature we incorporate into the model is that, instead of each paying c to intervene, a volunteer's cost decreases as more players choose to volunteer. We analyze the implications of this new cost function on the mixed-strategy and on the pure-strategy Nash equilibria, and then we conduct a welfare comparison of the two different types of equilibria as we did in Diekmann's original model, and show a major reversal of the comparison results with respect to the standard dilemma.

Cost-sharing is not a new idea in the study of the volunteer's dilemma. In general, the idea that cost-sharing will alleviate one's burden to comply with social morality in large groups and thus encourage cooperation is quite an intuitive one. These effects turned out to be the case for the two cost structures proposed in Weesie and Franzen (1998), namely evenly splitting the cost among volunteers as one and randomly drawing one volunteer to bear the full cost as another. Both of these schemes lead to an increase in the probability of (at least) one player volunteering compared to a game where the cost is not shared; however, the comparative statics, i.e., how the probabilities change with group size, remain qualitatively the same as in Diekmann's original model. In addition, the theoretical prediction is inconsistent with their experimental results given in the same paper: Indeed, despite the limited variation of n in the experiment ($n = 2, 4, 8$), "the observed macro probabilities of public goods production are almost always larger in the 8-person groups than in the smaller groups." The inconsistency with experimental results continues with more recent experimental studies on the subject; see for instance Goeree et al. (2017) and Campos-Mercade (2021).

Cost-sharing is a plausible assumption in the dilemma context. Weesie and Franzen (1998) provided several justifying contexts for their cost-sharing assumption: the chance of being hurt by the perpetrator is lower when more passers-by intervene; one volunteer needs to exercise less labor for the maintenance of a community center when more volunteers show up; the tension of complaining about someone's violation of public rule (e.g., talking in quiet-study areas) is lessened if other people join in the complaints...

Is Weesie and Franzen's (1998) theoretical prediction robust to other forms of cost-sharing? It turns out that the key comparative statics results depend on *how the cost is shared* among volunteers. In what follows, we propose a new cost structure that generates equilibrium comparative statics that may account for the inconsistency exhibited in the "macro probabilities of public goods production."

3.1 The model with a new cost structure

In this subsection, we propose a new form of cost-sharing for the volunteer's dilemma, a cost function that declines exponentially as the number of volunteers grows. As a preview, the main result is that this cost structure may reverse the comparative statics of the mixed-strategy equilibrium as far as the probabilities of participation are concerned.

For simplicity and without loss of generality, we normalize the benefit $b = 1$, and thus let the cost satisfy $0 < c < 1$, so that the key inequality that $c < b$ is maintained. When

k volunteers come forward, Weesie and Franzen (1998) assume that they split the cost c so that each volunteer incurs cost $\frac{c}{k}$. Instead, we shall assume that each volunteer incurs cost c^k. Both are simple and tractable specifications of the cost-sharing idea, in addition to being economically plausible and intuitive cost structures.[6]

Assumption. *If there are k volunteers, $1 \leq k \leq n$, then the cost to each volunteer for choosing V is c^k, where $0 < c < 1$.*

We now characterize the Nash equilibrium set for the game with this new cost structure. We first observe that the set of pure-strategy equilibria is the same as for the classical model. Indeed, if $(n-1)$ players do not volunteer, then it is a best response for the n^{th} player to volunteer since $c < b$. But if any one player volunteers, then no other player will do so, since this will be costly to her while the benefit is already there with one volunteer (and will not increase). Thus the mutual best responses pose a free-riding problem.

Observe that our cost function and the classical one coincide, i.e., $c^k = \frac{c}{k}$, only when $k = 1$, and $c^k < \frac{c}{k}$ when k grows sufficiently large (almost immediately for small c's, such as for $c < 0.5$). Hence, our assumption implies extra efficiency in production associated with cooperation, e.g., the individual risk of getting hurt drops by more than two-thirds if three passers-by jointly intervene in public violence, one volunteer needs to exercise less than half the effort in community maintenance as before if he collaborates with another volunteer, etc. Thus this cost function embodies a stronger form of cost sharing, one that exhibits the key property of increasing returns to group size.[7]

Imposing symmetry in equilibrium, the expected payoffs associated with the two pure strategies are

$$u(V \mid p) = 1 - \sum_{k=0}^{n-1} c^{k+1} C_{n-1}^{k} p^k (1-p)^{n-1-k} \text{ and } u(S \mid p) = 1 - (1-p)^{n-1}$$

where p is the (symmetric) probability of choosing V for each player, index k (of summation) is the number of other volunteers out of $(n-1)$ players except the focal player, C_{n-1}^{k} is the combinatorial number for selecting k active players out of $(n-1)$ players.

The tractable nature of our cost function is reflected in the fact that

$$u(V \mid p) = 1 - c\sum_{k=0}^{n-1} C_{n-1}^{k}(cp)^{k}(1-p)^{n-1-k} = 1 - c(cp + 1 - p)^{n-1},$$

which is a direct consequence of the binomial theorem formula.

For the mixed-strategy equilibrium, by equating the two expected payoffs $u(V \mid p)$ and $u(S \mid p)$, we can solve for the equilibrium probability of volunteering p^* of each individual, and then derive the probability of the collective good being produced $P^* = 1 - (1 - p^*)^n$ accordingly. Some simple algebra gives rise to the next result.

Theorem 2. *Under the exponential cost structure, there is a unique mixed-strategy equilibrium where the individual probability of volunteering p^* and the probability of the collective good being produced P^* are given by:*

$$p^* = 1 - \frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1} \text{ and } P^* = 1 - \left(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1}\right)^n. \tag{1}$$

The proof of this Theorem follows from the steps described in the preceding paragraph and thus the details are omitted. The tractability of the analysis here relies on the use of the binomial formula and thus a priori need not extend beyond the hyperbolic/exponential functional form.[8] Therefore, although there are other possible ways to model a qualitatively similar cost structure, tractability confines us to this specific exponential form, which is, nonetheless, very plausible and sufficient to generate a broader, novel and intuitive comparative statics pattern.

Theorem 3. *The equilibrium probabilities have the following properties.*

1. *Both p^* and P^* globally decrease in c.*
2. *p^* globally decreases in n.*
3. *There exist $0 < c_1, c_2 < 1$ such that:*

   i) when $0 < c < c_1$, P^* increases in n for all $n \geq 3$.[9]

   ii) when $c_1 < c < c_2$, P^* decreases in n first and then increases in n.

   iii) when $c_2 < c < 1$, P^* decreases in n for all $n \geq 3$.

Compared to Weesie and Franzen (1998), who arrived at the conclusion that P^* is globally decreasing in n for all parameter ranges, Theorem 9 allows P^* to increase almost globally in n for sufficiently small c. On the other hand, when c is sufficiently large (as in 3 (iii)) the two models' conclusions coincide. For intermediate values of c, P^* is inverse-U-shaped in n, in line with it being an in-between scenario of the other two cases.

Therefore, our results provide a potential justification for the lack of congruence with the theory that was found in Weesie and Franzen's (1998) experimental data. Put in a realistic context, the Theorem suggests that given the exponential cost structure, whether a larger crowd favors the production of the collective good depends on the magnitude of the cost parameter of volunteering (i.e., c). That is, if the perpetrator appears to be weak and easily restrained, the violence is more likely to be prevented or stopped when there are more witnesses. Similarly, if the work load of the community center's maintenance is relatively light, the center is more likely to be well maintained with a larger community size. The opposite holds if the perpetrator appears ferocious (or is armed) or if the workload seems daunting, corresponding to a high volunteering cost.

In addition to the absolute magnitude of the cost, the comparison of our model and Weesie and Franzen's (1998) model suggests that the individual and collective incentives for volunteering are also affected by the way the costs are shared among all volunteers. As mentioned earlier, the exponential cost structure is stronger than the even-splitting cost structure, and consequently, such extra efficiency in cooperation makes the collective good more likely to be produced in a larger group.

Naturally, the mixed-strategy equilibrium in our setting still reflects excessive volunteering and cost duplication with a significant probability. Nonetheless, the strong cost-sharing structure implies that the payoff cost to such mis-coordination remains bounded, so that the probability of the public good being provided ends up being higher with larger group sizes.

3.2 Welfare comparison with cost sharing

In the classical version of the volunteer's dilemma, we have conducted welfare comparison for the pure-strategy and mixed-strategy equilibria, and confirmed that $\tilde{W}_{PS} > \tilde{W}_{MS}$ always holds as the former avoids any duplication of volunteering costs by ensuring sure success with a single volunteer in equilibrium.

The same comparison issue can be addressed here, with our form of cost sharing, but now one needs a fresh look at the question because of the advantage of cost savings when two or more players volunteer.

As observed earlier, despite the cost-sharing efficiency, the pure-strategy social welfare is the same as in the classical model, because only one player volunteers in any equilibrium. Therefore, the utilitarian welfare corresponding to any of the equivalence class of pure-strategy Nash equilibria is given by the total benefit minus the total cost, i.e.,

$$W_{PS} = \tilde{W}_{PS} = n - c.$$

For the (unique) mixed strategy equilibrium, the expected social welfare is

$$\begin{aligned} W_{MS} &= \sum_{k=1}^{n} (n - kc^k)\Big(C_n^k p^k (1-p)^{n-k}\Big) \\ &= n - n(1-p)^n - \sum_{k=1}^{n} k\Big(C_n^k (cp)^k (1-p)^{n-k}\Big) \\ &= n - n(1-p)^n - \sum_{k=1}^{n} ncp\Big(C_{n-1}^{k-1} (cp)^{k-1} (1-p)^{n-k}\Big) \\ &= n - n(1-p)^n - ncp(1 + cp - p). \end{aligned}$$

where $\Big(C_n^k p^k (1-p)^{n-k}\Big)$ is again the binomial probability of k players volunteering in the equilibrium, while $(n - kc^k)$ is the corresponding social welfare with cost-sharing (reflected in c^k as a decreasing function of k). Recall the equilibrium probability is solved in (1) to be

$$p = p^* = 1 - \frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1}.$$

Given the complex expression of W_{MS} once p is substituted in, and its lack of tractability, we need to resort to graphical simulation to shed light on W_{MS} and W_{PS} as functions of n and c for a magnitude comparison. After simulating the difference $W_{PS} - W_{MS}$ for sufficiently large range of c and n, we found the results to suggest that $W_{PS} > W_{MS}$ in most cases, but we do have $W_{PS} < W_{MS}$ for sufficiently small c (approximately for $c < 0.38$) and small n as well, when the cost-sharing effect is expected to be rather strong.

In other words, the mixed-strategy equilibrium may dominate the pure-strategy equilibrium in terms of expected total welfare. This is strong reversal outcome for a coordination game since the reverse conclusion is broadly the norm for a large class of coordination games, including the standard volunteer's dilemma. Intuitively, this conventional conclusion is due to the fact that the mixed-strategy equilibrium involves a substantial amount of irrelevant volunteering and associated cost duplication, while the pure-strategy equilibrium is socially efficient, as it avoids any such excessive volunteering and cost duplication.[10]

In order to provide some intuition, let us inspect the two-player case, i.e., $n = 2$, and let $f(c) = W_{PS} - W_{MS}$. Then we have $f(0) = 0$, and

$$f'(c) = -\frac{c^4 - 2c^3 + 5c^2 - 6c + 1}{(c^2 - c + 1)^2}.$$

Moreover, since $f'(0) = -1$ and $f'(1) = 1$, this means that the polynomial in the numerator has at least one real root in the interval $(0, 1)$. Therefore, there exists a $\bar{c} \in (0, 1)$ for which f is decreasing for $c \in (0, \bar{c})$, and since $f(0) = 0$, then we get that $f(c) < 0$ for $c \in (0, \bar{c})$. As a result for $n = 2$, there exists an interval $(0, \bar{c})$ for which $W_{PS} < W_{MS}$.

In the general case, for $n > 2$, we have, with $A = \left(\frac{1}{c}\right)^{\frac{1}{n-1}}$

$$\begin{aligned} f'(c) = \frac{n^2\,(nA - n + 1)}{(n-1)(c + A - 1)^2}\left(\frac{c}{c + A - 1}\right)^{(n-1)} &- \frac{c^3 n^2 A^2 (nA - N + 1)}{(n-1)(c + A - 1)^5} \\ &- \frac{cn(A-1)(c - nA + n - 2 + A)}{(n-1)(c + A - 1)^2} - 1. \end{aligned}$$

We observe that, for each n, we can identify a $\bar{c}$ (the larger n, the smaller $\bar{c}$) such that $f'(c) < 0$ for $c \in (0, \bar{c})$. Given $f(0) = 0$, in the same spirit, it follows that we have $f(c) < 0$ for $c \in (0, \bar{c})$. Therefore, with sufficiently small c and n, the mixed-strategy equilibrium, despite its inherent cost duplication, can generate higher social welfare than the pure-strategy equilibrium, as a result of the proposed cost-sharing rule. This is a novel reversal to the standard welfare comparison results for many such anti-coordination games. In our case, the cost duplication is partially offset by the cost-sharing efficiency among volunteers, especially when the individual cost is sufficiently low that the exponential decay of costs is thus strong.

4 Conclusion

This paper has re-examined the volunteer's dilemma, with as principal motivation, to explore a plausible modification of the cost structure involving cost sharing with a view to inducing the mixed-strategy equilibrium to reflect that a larger group size may favor the production of the collective good. To this end, the key ingredient is a novel (individual) cost function that decays exponentially in the number of volunteers. This is a plausible and realistic scenario in some practical contexts for the volunteer's dilemma game, including some of the well-known common applications of the model. In addition, this theoretical finding may partially explain the divergence between the predictions of the classical theory and associated experimental results presented in many past studies.

Our result suggests that the characteristics of the individual and the collective incentives to volunteer are not only affected by the absolute magnitude of the volunteering cost but also by the way the cost is shared among all volunteers, i.e., how efficiently the volunteers may collaborate with each other. In other words, it appears that a key property of the present cost function is the increasing returns to group size implicit in the exponential cost structure, which is interactively affected by the magnitude of the volunteering cost.

Furthermore, the cost structure is also analytically tractable, which is of course a natural requirement, since the attending computational complexity for mixed-strategy equilibrium and its comparative statics properties could easily turn out to be prohibitive. The main result of this paper establishes that this choice of cost function essentially delivers our desired conclusions on the collective production probability for the public good. Another result is that the mixed-strategy equilibrium is more socially efficient than the pure-strategy equilibrium for a significant range of parameter values. This in turn is a major departure from the standard volunteer's dilemma as well as from well-known conclusions for coordination games more broadly.

Finally, further work on the topic may aim at strengthening our results by selecting other meaningful and more powerful forms of cost sharing. However, due to the inherent combinatorial aspects of the game, it seems hard to further generalize the cost structure without compromising analytical tractability.

References

Amir, R. (2005). Supermodularity and complementarity in economics: An elementary survey. *Southern Economic Journal 71*(3), 636–660.

Archetti, M. (2009). The volunteer's dilemma and the optimal size of a social group. *Journal of Theoretical Biology 261*(3), 475–480.

Archetti, M. and I. Scheuring (2011). Coexistence of cooperation and defection in public goods games. *Evolution 65*(4), 1140–1148.

Bliss, C. and B. Nalebuff (1984). Dragon-slaying and ballroom dancing: The private supply of a public good. *Journal of public Economics 25*(1-2), 1–12.

Campos-Mercade, P. (2021). The volunteer's dilemma explains the bystander effect. *Journal of Economic Behavior & Organization 186*, 646–661.

Diekmann, A. (1985). Volunteer's dilemma. *Journal of conflict resolution 29*(4), 605–610.

Goeree, J. K., C. A. Holt, and A. M. Smith (2017). An experimental examination of the volunteer's dilemma. *Games and Economic Behavior 102*, 303–315.

Konrad, K. A. (2024). The collective security dilemma of preemptive strikes. *European Journal of Operational Research 313*(3), 1191–1199.

Konrad, K. A. and F. Morath (2021). The volunteer's dilemma in finite populations. *Journal of Evolutionary Economics 31*(4), 1277–1290.

Manning, R., M. Levine, and A. Collins (2007). The kitty genovese murder and the social psychology of helping: The parable of the 38 witnesses. *American Psychologist 62*(6), 555.

Weesie, J. (1993). Asymmetry and timing in the volunteer's dilemma. *Journal of conflict resolution 37*(3), 569–590.

Weesie, J. (1994). Incomplete information and timing in the volunteer's dilemma: A comparison of four models. *Journal of conflict resolution 38*(3), 557–585.

Weesie, J. and A. Franzen (1998). Cost sharing in a volunteer's dilemma. *Journal of conflict resolution 42*(5), 600–618.

Notes

1. Archetti (2009) discusses animal environments in which the dilemma emerges, such as when the choice is whether to make alarm calls that are costly for the call maker, but alert the larger group to a potential predator.

2. While this version of events was reported by The New York Times and widely accepted at the time as the sad truth behind the story, later evidence surfaced to contradict this account (see Manning et al., 2007).

3. A different approach is followed by Archetti and Scheuring (2011) who study the volunteer's dilemma in a case where the cost structure is the standard one, but the group benefit emerges only if multiple players volunteer.

4. Theoretically, we can generalize the requirement on the number of volunteers needed for production to, say, that $m \geq 1$ volunteers are needed for production. However, this parameter turns out to affect the order of the equation in solving the mixed-strategy equilibrium and higher-order polynomial will render the equilibrium unsolvable in terms of an explicit (and manageable) solution.

5. As such, the game is one of strategic complements, or equivalently a supermodular game (see e.g., Amir, 2005).

6. Indeed, due to the mathematical properties of the game and the need to work with binomial expansions, we could not find another tractable way to model cost-sharing in this context except the hyperbolic function given in Weesie and Franzen (1998) and the exponential function studied here.

7. For instance, when $c = 0.4$, we have that, as k goes across the values 1,2, and 4, the cost declines from 0.4 to 0.16, to 0.0256 (instead of from 0.4 to 0.2, to 0.1 respectively, as implied by the cost-sharing form of Weesie and Franzen, 1998).

8. In particular, if we change the assumption to require more than one volunteer's participation in the good's production (e.g., the violence can only be stopped when two or more passers-by intervene), then the index k of the summation in $u(V \mid p)$ should start from a number greater than 0, and then the equation $u(V \mid p) = u(S \mid p)$ becomes a higher order polynomial of p which is unsolvable.

9. Notice that we impose $n \geq 3$ instead of $n \geq 2$. In fact, P^* always decreases in n at the point $n = 2$, but may start to increase when n is slightly larger than 2 so that discretely P^* is greater at $n = 3$ than at $n = 2$. To avoid mathematical complications we ignore P^*'s property at the single point $n = 2$.

10. Likewise, in the classical Battle of the Sexes, the mixed-strategy equilibrium is dominated by both of the pure-strategy equilibria (for both players), which is thus a more extreme outcome than for the volunteer's dilemma. Recall that in the Battle of the Sexes, the mixed-strategy equilibrium is welfare-inferior because independent randomization causes the two players to mis-coordinate, and thus get a zero payoff, with a significant probability.

5 Appendix

This section of the paper collects all the analytical proofs of the paper.

Proof of Theorems 1 and 2

Theorem 1 follows directly from equating $\tilde{W}_{PS} - \tilde{W}_{MS} = -c + n(1-p)^n + ncp$ and by replacing p with p^*, one gets $\tilde{W}_{PS} - \tilde{W}_{MS} = c(n-1) > 0$ for $c \in (0,1)$ and $n \geq 2$.

Theorem 2 follows by equating the expected payoffs of actions V and S, $u(V \mid p) = u(S \mid p)$, and then we can solve for the equilibrium probability p^* and P^* by their definitions.

Proof of Theorem 3

The proof of Theorem 3 contains three parts. First, we need to show that the individual volunteering probability and macro probability of success, p^* and P^*, both decrease in the cost c. Second, we need to show that the individual volunteering probability p^* decreases in the group size n (as in the standard model). Then follows the novel part of the results, that is, the macro probability of success P^* can exhibit non-monotonic patterns with respect to n, depending on the magnitude of c.

First, let us start by showing that both p^* and P^* decrease in c. Since $P^* = 1-(1-p^*)^n$, it suffices to show that p^* decreases in c, which follows immediately from the transformation:

$$p^* = 1 - \frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1} = 1 - \frac{1}{\frac{1}{c}\left(\left(\frac{1}{c}\right)^{\frac{n}{n-1}} - 1\right) + 1}.$$

Therefore, it is straightforward that p^* is a decreasing function of c, and then it follows that P^* (as a monotonic transformation of p^*) also decreases in c. In the same expression, one can easily check that p^* is also a decreasing function of n (given that $c < 1$).

Now that we have proved the first and second arguments of Theorem 3, we can proceed with the more important comparative statics of P^* with respect to n. Differentiating P^* with respect to n in Eq. (1) yields

$$\frac{dP^*}{dn} = -\left(\frac{1}{\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c}}\right)^n \left(\frac{n\left(\frac{1}{c}\right)^{\frac{n}{n-1}} \log(\frac{1}{c})}{(n-1)^2\left(\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c}\right)} + \log\left(\frac{1}{\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c}}\right)\right).$$

Notice that $\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c} = \frac{1}{c}\left(\left(\frac{1}{c}\right)^{\frac{1}{n-1}} - 1\right) + 1 > 1$ because $\frac{1}{c} > 1$, thus $\left(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1}\right)^n$ is positive. For the second bracketed term, let us define

$$f(n) = \frac{n\left(\frac{1}{c}\right)^{\frac{n}{n-1}} \log(\frac{1}{c})}{(n-1)^2\left(\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c}\right)} \text{ and } g(n) = -\log\left(\frac{1}{\left(\frac{1}{c}\right)^{\frac{n}{n-1}} + 1 - \frac{1}{c}}\right), \tag{2}$$

so the second bracketed term equals $f(n) - g(n)$. Now it is clear that $f(n) > 0$ and $g(n) > 0$, and the sign of $\frac{dP^*}{dn}$ depends on the sign of $-\big(f(n) - g(n)\big)$ or $g(n) - f(n)$. To sign $\frac{dP^*}{dn}$, we need to study the properties of $g(n)$ and $f(n)$. It is straightforward to see that $g(n)$ decreases in n, but other than this, it is hard to sign $f'(n)$, $f''(n)$ and $g''(n)$ analytically. However, since the only parameter is c, which ranges between 0 and 1, using simulation, we have confirmed that both $g(n)$ and $f(n)$ are decreasing convex functions of n for all $0 < c < 1$.

We first show that there exists some c_1, supposedly to be a small number close to 0, such that when $0 < c < c_1$, $f(n) < g(n)$, so that P^* increases in n for all $n \geq 3$. It is hard to (analytically) directly solve the inequality and find c_1, but we can inspect the values of $f(n)$ and $g(n)$ when $c \to 0$. Indeed, we can show that $f(n) < g(n)$ for all $n \geq 3$ when $c \to 0$ by the following arguments.

First note that when $c \to 0$, $g(n) \to -\log(\frac{0}{\infty - 1}) = -\log(0) = \infty$, while

$$f(n) = \frac{n}{(n-1)^2} \frac{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1} \log(\frac{1}{c}) \to \infty.$$

Thus we can use L'Hospital's Rule to prove $\frac{f(n)}{g(n)} < 1$ when $c \to 0$. Transforming the inequality, $\frac{f(n)}{g(n)} < 1$ is equivalent to

$$\frac{n}{(n-1)^2} \left(\frac{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1} \right) \left(\frac{\log(\frac{1}{c})}{-\log(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1})} \right) < 1.$$

When $c \to 0$, $\left(\frac{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1} \right) \to 1$, while both $\log(\frac{1}{c}) \to \infty$ and $-\log(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1}) \to \infty$. Using L'Hospital's Rule twice for the second bracketed term here, one gets:

$$\frac{\log(\frac{1}{c})}{-\log(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1})} \to \frac{n - 1 - (c)^{\frac{n}{n-1}}\,(n-1)^2}{n} \to \frac{n-1}{n}.$$

Thus

$$\frac{f(n)}{g(n)} \to \frac{n}{(n-1)^2} \frac{n-1}{n} = \frac{1}{n-1} < 1, \quad \text{when } c \to 0, \text{ for all } n \geq 3.$$

Now that $f(n) < g(n)$ holds strictly for all $n \geq 3$ when $c \to 0$, and also $f(n)$ and $g(n)$ are continuous in the parameter c for all $0 < c < 1$, there must exist some $c_1 > 0$, supposedly small, such that $f(n) < g(n)$ for all $0 < c < c_1$. The proof is now complete for this case.

Next, we show that there exists some $c_2 > c_1$ such that for $c_1 < c < c_2$, there exists some $\tilde{n}$ such that $f(n) > g(n)$ for $n < \tilde{n}$ and $f(n) < g(n)$ for $n > \tilde{n}$, so P^* decreases in n first and increases in n thereafter. We need three steps. First, we can show $f(2) > g(2)$ to be always true. Second, we can show $f(n), g(n) \to 0$ when $n \to \infty$. Third, we want to show that $f(n)$ and $g(n)$ cross exactly once when n increases from 2 to infinity.

To show the third step, we can do the following. We can show $f'(2) < g'(2)$ and $f'(\infty) > g'(\infty)$ when c is sufficiently small. Then since $f(n)$ and $g(n)$ are smooth convex functions, their derivatives $f'(n)$ and $g'(n)$ must cross at least once when n increases from 2 to infinity.

Suppose they last cross at n^*, which implies $0 > f'(n) > g'(n)$ for all $n > n^*$. Now that $f(\infty) = g(\infty) = 0$, and g decreases faster than f for all $n > n^*$, it must be that $f(n) < g(n)$ for all $n \geq n^*$. Now that we have $f(2) > g(2)$, $f(n) < g(n)$ for all $n \geq n^*$, and $f(\infty) = g(\infty) = 0$, combined with the fact that f and g are smooth convex functions, we can conclude that f and g cross only once at some $\tilde{n}$ smaller than n^*. And the proof should be complete for this case. Therefore, we need to prove the following: (i) $f(2) > g(2)$; (ii) $f(\infty) = g(\infty) = 0$; (iii) $f'(2) < g'(2)$ and $f'(\infty) > g'(\infty)$ when c is sufficiently small. Now let us do it step by step.

(i) $f(2) > g(2)$, i.e., we want to show $\frac{2\left(\frac{1}{c}\right)^2 \log(\frac{1}{c})}{\left(\left(\frac{1}{c}\right)^2 + 1 - \frac{1}{c}\right)} > -\log(\frac{1}{\left(\frac{1}{c}\right)^2 + 1 - \frac{1}{c}})$. Let $a = \frac{1}{c} > 1$. We want to show $\frac{2a^2 \log(a)}{(a^2+1-a)} > -\log(\frac{1}{a^2+1-a}) = \log(a^2+1-a)$, that is, $2a^2\log(a) - (a^2+1-a)\log(a^2+1-a) > 0$, that is, $a^2\log(a^2) - (a^2+1-a)\log(a^2+1-a) > 0$. Notice the function $x\log(x)$ is increasing in x. Since $a = \frac{1}{c} > 1$, so $a^2 > a^2 + 1 - a$, and thus $a^2\log(a^2) > (a^2+1-a)\log(a^2+1-a)$, and the inequality is proved.

(ii) $f(\infty) = g(\infty) = 0$. When $n \to \infty$, since $\left(\frac{1}{c}\right)^{\frac{1}{n-1}} \to \left(\frac{1}{c}\right)^0 = 1$,

$$f(n) = \frac{n\left(\frac{1}{c}\right)^{\frac{1}{n-1}} \log(\frac{1}{c})}{(n-1)^2\left(\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1\right)} \to 0,$$

$$g(n) = -\log(\frac{c}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1}) \to -\log(\frac{c}{c}) = 0.$$

(iii) $f'(2) < g'(2)$ and $f'(\infty) > g'(\infty)$ when c is sufficiently small. Since $f' < 0$ and $g' < 0$, we want to show $\frac{f'(2)}{g'(2)} > 1$ and $\frac{f'(\infty)}{g'(\infty)} < 1$.

$$\frac{f'(n)}{g'(n)} = \frac{n^2 - 1 + \frac{(c-1)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1} n}{(n-1)^2}.$$

So

$$\frac{f'(n)}{g'(n)} > / < 1 \iff n^2 - 1 + \frac{(c-1)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1} n > / < (n-1)^2 \iff$$

$$2n - 2 > / < \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1} n \iff n\left(2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}} + c - 1}\right) > / < 2.$$

When $n = 2$, we can show $\frac{(1-c)\log(\frac{1}{c})}{\frac{1}{c}+c-1} < 1$ for all c, so $2\left(2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)+c-1}\right) > 2$ and thus $\frac{f'(2)}{g'(2)} > 1$ for all c. The proof goes as follows. Let $b = \frac{1}{c} > 1$, then $\frac{(1-c)\log(\frac{1}{c})}{\frac{1}{c}+c-1} = \frac{\left(1-\frac{1}{b}\right)\log(b)}{b+\frac{1}{b}-1}$. We want to show $\frac{\left(1-\frac{1}{b}\right)\log(b)}{b+\frac{1}{b}-1} < 1 \iff \left(1-\frac{1}{b}\right)\log(b) < b+\frac{1}{b}-1 \iff (b-1)\log(b) < b^2-b+1 = b(b-1)+1$. Notice $\log(b) < b$ for all $b > 1$, thus $(b-1)\log(b) < b(b-1) < b(b-1)+1$, and $\frac{(1-c)\log(\frac{1}{c})}{\frac{1}{c}+c-1} < 1$ is proved. Thus we have shown $f'(2) < g'(2)$ for all c.

Now examine the case of $n \to \infty$. Recall $\frac{f'(n)}{g'(n)} < 1$ if $\left(2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right)n < 2$, and when $n \to \infty$, the LHS goes to either ∞ or $-\infty$ depending on the sign of the constant $\left(2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right)$. Since $\left(\frac{1}{c}\right)^{\frac{1}{\infty}} \to 1$, so $2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1} \to 2 - \frac{(1-c)\log(\frac{1}{c})}{c}$, and thus we need to examine the sign of this expression with respect to c. Notice that the derivative of $\frac{(1-c)\log(\frac{1}{c})}{c}$ with respect to c is $\frac{-1+c-\log(\frac{1}{c})}{c^2} < 0$, so it decreases in c. When $c \to 0$, $\frac{(1-c)\log(\frac{1}{c})}{c} \to \frac{\log(\infty)}{0} = \infty$. When $c \to 1$,$\frac{(1-c)\log(\frac{1}{c})}{c} = 0$. Due to monotonicity, there exists some $c_2 > 0$ such that $\frac{(1-c_2)\log(\frac{1}{c_2})}{c_2} = 2$. And for all $c < c_2$, $\frac{(1-c)\log(\frac{1}{c})}{c} > 2$, so $\lim_{n\to\infty}\left(2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right) < 0$ and thus $\left(2 - \frac{(1-c)\log(\frac{1}{c})}{\left(\frac{1}{c}\right)^{\frac{1}{n-1}}+c-1}\right)n \to -\infty$, so $\frac{f'(\infty)}{g'(\infty)} < 1$, or $f'(\infty) > g'(\infty)$. Vice versa, for all $c > c_2$, $\frac{f'(\infty)}{g'(\infty)} > 1$, or $f'(\infty) < g'(\infty)$.

Now that for $c < c_2$, we have shown (i)(ii)(iii) for $c < c_2$. Still, we need to verify that $c_1 < c_2$. Recall that c_1 is a number any close to zero such that when $0 < c < c_1$, we have $f(n) < g(n)$ for all $n \geq 3$ given that (1) $\lim_{c\to 0} f(n;c) < \lim_{c\to 0} g(n;c)$ for all $n \geq 3$ and (2) $f(n;c), g(n;c)$ are continuous in c. Therefore, we only need to verify that $f(n) > g(n)$ for all $n \geq 3$ at c_2, and then by definition $c_1 < c_2$. Recall c_2 is defined implicitly by $\frac{(1-c_2)\log(\frac{1}{c_2})}{c_2} = 2$, which can be solved by numerical approximation to be $c_2 \approx 0.3464$, and by simulation one can easily see that $f(n) > g(n)$ for all $n \geq 3$ at $c_2 \approx 0.3464$, so it follows that $c_1 < c_2$. Lastly, note that the proof stated above leads to the conclusion that P^* decreases in n first and increases in n thereafter for all $c < c_2$, including those $c < c_1$, but this does not conflict with the first case. The reasoning we followed so far points to the intuition that for P^* to exhibit an increasing interval with respect to n, a smaller c is more favorable. In fact, when $c < c_1 < c_2$, it can be seen by simulation that the threshold dividing the decreasing and increasing parts of $P^*(n)$ is less than 3, meaning P^* only decreases before $n = 3$ and increases for all $n \geq 3$, so there is no conflict between 3.(i) and 3.(ii) of Theorem 9.

Lastly, we need to prove the third case, i.e., that P^* decreases for all $n \geq 3$ when $c > c_2$. In the second case, we have already proved that $f(2) > g(2)$, $f(\infty) = g(\infty) = 0$, $f'(2) < g'(2)$ for all c, and $f'(\infty) < g'(\infty)$ when $c > c_2$ (the other case). For smooth convex functions $f(n)$ and $g(n)$ it is only possible that $f(n) > g(n)$ for all $n \geq 2$ (which is confirmed by simulation), though in Theorem 9 we relaxed it to $n \geq 3$ for consistency. So 3.(iii) is proved, and the proof is complete. Q.E.D