Comparative Statics for the Private Provision of Normal and Inferior Public Good

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Abstract

This paper studies a standard model of the private provision of public goods under the assumption that the public good is either normal or inferior for every consumer at every level of wealth. Using new tools from monotone comparative statics, we show that the condition of normality (inferiority) of public good is sufficient for the extremal total equilibria contributions to be increasing (decreasing) with the number of consumers. The lattice-theoretic methodology we use also allows us to generalize the classic existence result by showing that the assumption of quasi-concavity of the utility function is not "critical" and therefore can be relaxed.

Keywords: Private provision of public goods, monotone comparative statics, supermodular games, group size.

Jel codes: H41, L13

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1 Introduction

The theory of voluntary provision of public goods has received growing attention after the publication of Samuelson's (1954; 1955) seminal works. Since then the theory has been discussed in Olson (1965), McGuire (1974), Laffont (1988), Warr (1982, 1983), Bergstrom et al. (1986), Varian (1994), Kerschbamer and Puppe (1998), Gaube (2000, 2001) and others. One of the most striking results was obtained in Bergstrom et al. (1986), where the condition of normality of both private and public goods for every individual in the economy was proved to be sufficient for the uniqueness of Nash equilibrium. Important contributions to the theory have also been made by Cornes and Sandler (1984a,b, 1994, 1996). They noticed that Nash equilibrium is generally not efficient (in the sense that contribution to the amount of public good according to Nash equilibrium outcome is less than the one which is Pareto efficient). Since then much attention has been engaged to the search for possible solutions that would overcome under-provision (which is considered as a sign of market failure and a justification for government intervention).

Many works have also been concerned with the problem of free-riding (or easy riding, as Cornes and Sandler call it, since the individuals contribute less rather than not contribute at all) and the exacerbation of this tendency as the group size increases. It is a widely accepted hypothesis that when the public good is voluntarily provided, incentives to free ride increase with the number of individuals (Olson, 1965; Laffont, 1988; Mueller, 1989; Sandler, 1992). But this claim has been usually illustrated only by means of examples of Cobb-Douglas and quasilinear utility functions. Gaube (2001) found that sufficient conditions for this general presumption to hold are conditions for both public and private goods to be strictly normal and weak gross substitutes.

Another important issue in the theory of public goods is whether a sequential model of private provision leads to greater or lesser contribution than a simultaneous move game. Varian (1994) studies both models and shows that under the condition that the public good is normal, the total contribution to the public good in a sequential game is never larger than in a simultaneous-move game.

All in all, the assumption of normality of both goods for all consumers is easily seen to be standard in the theory of private provision of public goods. The purpose of the present paper is to see whether this assumption, widely used in the theory, allows us to derive unambiguous comparative statics conclusions. In addition, the public good in many scenarios may be better characterized by its inferiority instead of normality (e.g., some public parks are mostly utilized by the homeless). So the second part of this paper also considers a novel case of an inferior public good and a normal private good. The paper provides a thorough comparative statics analysis for the traditional model of voluntary provision of public goods when it is normal and when it is inferior. More precisely, we investigate the question of how the equilibria outcomes, namely equilibrium per-consumer consumption of the private good, equilibrium per-consumer contribution to the public good, and the total equilibrium supply of the public good change when the group size increases.

The comparative statics analysis presented in this paper relies on a new monotone comparative statics approach based on lattice-theoretic methods. This approach is widely known as having advantages over traditional methods based on the Implicit Function Theorem and signing derivatives. Lattice-theoretic methods were developed by Topkis (1978, 1979), Vives (1990), further analyzed by Milgrom and Roberts (1990, 1994), Milgrom and Shannon (1994), Amir and Lambson (2000), Amir (1996b, 2003). Its main feature consists of utilizing only a subset of the assumptions needed for the standard approach which, as a result, demonstrates that not all traditional assumptions are 'critical' for the comparative statics. It also helps to understand the economic meaning rather than to be focused on the mathematical assumptions. Furthermore, compared to the traditional approach, it does not give rise to statements that are not always well-defined (by dealing only with the extremal equilibria).

The approach we use leads to unambiguous, meaningful statements about comparative statics issues. It also allows for some generalization of the existing results in the literature on public goods. For example, sufficient conditions for the uniqueness of Nash equilibrium (Bergstrom et al., 1986) and the exacerbation of free riding (Gaube, 2001) were obtained under the standard assumption of quasi-concavity of the utility function. Applying new tools from lattice-theoretic methodology, we are able to obtain these results without necessarily imposing this assumption. We show that this assumption is not 'critical' and can be relaxed and these results will still be valid.

Most of the theoretical and experimental work on public goods normally deals with a simple model in which a single pure public good is supplied and consumers are concerned only about their private consumption and the total supply of the public good. We use a simple version of this model due to Bergstrom et al. (1986). The economy has n consumers and two goods (one public and one private). We first generalize the Nash equilibrium existence result from Bergstrom et al. (1986), which was obtained under the assumptions that both public and private goods are normal and the utility function is strictly quasi-concave. We show that the assumption of normality of public good is sufficient to prove the existence. Then we analyze how the equilibria variables change with the number of consumers. It turns out that the normality of the public good is a sufficient condition for the total equilibrium contribution to the public good to be increasing, whereas for the individual contribution to the public good to be decreasing with n, we need to assume the normality of the private good as well. Indeed, the normality or inferiority of the private good is the key determinant for the individual contribution to be decreasing or increasing in n, thus a key factor to affect the free-riding incentives. On the other hand, the normality or inferiority of the public good determines the comparative statics of the total contribution in n, thus related to the overall success or failure of the private provision of public goods.

The rest of the paper is organized as follows. We present the model in Section 2. The case with normal public good is analyzed in Section 3, and the case with inferior public good is analyzed in Section 4. Section 5 concludes. All proofs are contained in the Appendix.

2 The Model

This section describes a simple model of the private provision of public goods with one public good, one private good, and n consumers. The main question is how the equilibria variables change with the number of consumers. The approach we use to answer the question is based on fundamental results from supermodular theory. First, we introduce some basic notation.

Each consumer i consumes an amount x_i of private good and contributes an amount of $y_i \geq 0$ to the supply of public good. The total contribution of all consumers is denoted by z, $z = \sum_{i=1}^{n} y_i$, which is used for the production of the public good. The output of the public good is then given by $c = f(z)$, where $f(z)$ can be interpreted as a production function. The utility function of consumer i is $u_i(x_i, c)$, who is endowed with wealth w. We assume that utility is twice-continuously differentiable. The price of the private good is p, whereas the price of the public good is normalized to 1. The sum of contributions to the public good by all consumers other than i will be denoted by y_{-i}. Given the voluntary contributions y_{-i} of other consumers, consumer i solves the maximization problem for utility function $u_i(x_i, c)$.

Definition 1. *A Nash Equilibrium for the model of private provision of the public good is a vector of contributions $y_i^*, i = 1, ..., n$, such that for each i, (x_i^*, y_i^*) is a solution to the following utility maximization problem*

$$\max_{x_i, y_i} U^i(x_i, c)$$

$$s.t. \;\; px_i + y_i = w$$

$$c = f(y_i + y_{-i}^*)$$

$$x_i \geq 0, \;\; y_i \geq 0$$

In a game with n consumers, we use the upper script n to denote the equilibrium level of variables. That is, let x_i^n be the equilibrium level of consumption of the private good for consumer i, y_i^n be the equilibrium level of contribution to the public good for consumer i, etc. When there is no risk of confusion, we use the same notations, x_i^n, y_i^n, y_{-i}^n, z^n, etc., to denote the equilibria set of variables, if the uniqueness of equilibrium is not guaranteed. In such cases, we use an upper and a lower bar in the notation of equilibria variable sets to denote the maximal and minimal elements of the sets, i.e., the maximal and minimal equilibria, correspondingly. We aim to predict the direction of changes in these extremal equilibria as the number of consumers changes. In addition, let $V_i(w, y_{-i})$ be the indirect utility function of consumer i, given wealth w and other consumers' contribution y_{-i}, where

$$V_i(w, y_{-i}) = \max_{0 \leq y_i \leq w} U^i(\frac{1}{p}(w - y_i), f(y_i + y_{-i})). \tag{1}$$

By choosing his own contribution y_i, a consumer is also choosing the total contribution $z =$

$y_{-i} + y_i$, given other $(n-1)$ consumers contributing y_{-i} in total. Therefore, the maximization problem can be rewritten as follows:

$$\max_{y_{-i} \le z \le w + y_{-i}} U^i\left(\frac{1}{p}(w + y_{-i} - z), f(z)\right). \tag{2}$$

The following assumptions are valid throughout the paper:

(A0) *$f(z)$ is strictly increasing.*

(A1) *$U^i(x_i, c)$ is twice continuously differentiable with $U_1^i, U_2^i > 0$.*

(A2) *(Normality of the private good) $U_2^i U_{21}^i - U_1^i U_{22}^i - \frac{f''}{(f')^2} U_1^i U_2^i > 0$.*

We note that the only requirements of the utility function are twice continuously differentiability (to use the Monotonocity Theorem in Milgrom and Shannon, 1994) and monotonicity. The utility function does not necessarily need to be quasi-concave. In fact, our main results are still valid with a convex production function $f(z)$ and U^i that is not quasi-concave. Under assumption (A2), the private good is normal (see Lemma 2 in the Appendix). However, this assumption is not needed in the case of a normal public good for our main results concerning the equilibrium existence and the comparative statics of the total contribution of the public good (Propositions 1 and 2), but is needed in this case to characterize the equilibrium individual contribution of the public good (Proposition 3), and is crucial for the general arguments with the inferiority of the public good. It is a plausible assumption given the fact that x_i is representative of an average private good, as the consumer only consumes two goods (the private and the public goods). In addition to the usual normality requirements on U^i (with respect to $U_2^i U_{21}^i - U_1^i U_{22}^i$), we need the production function f to be not too convex, which is naturally satisfied in view of f being a concave function in the usual case. The relaxation of this assumption, as will be discussed below, combined with the normality of the public good will give rise to each consumer's contribution of the public good to increase in n, and thus the consumption of the private good to decrease in n.

Remark. The results of the paper are applicable for a setting with n firms, where the production function of some firm i depends on a private input x_i of good 1 and some input z of good 2, with $z = y_i + y_{-i}$, so that each firm uses not only its private input y_i of good 2 but also what is available from others' input.

3 The case with the normality of the public good

In this section, we consider the case when the normality of the public good holds for every consumer i, $i = 1, ..., n$. We show that the normality is guaranteed by the following assumption (A3), which further ensures that the utility function (2) has the strict single-crossing property (Milgrom and Shannon, 1994) on (z, y_{-i}), thus implying that the extremal total equilibrium contributions to the public good, $\bar{z}^n$ and $\underline{z}^n$, are increasing in the number of consumers.

(A3) *(Normality of the public good)* $U_1^i U_{21}^i - U_2^i U_{11}^i > 0$.

The first proposition establishes the existence of a Nash equilibrium and that it is always symmetric.

Proposition 1. *With the normality of the public good (A3), there exists a symmetric Nash Equilibrium, and no asymmetric equilibria exist.*

The existence result obtained by Bergstrom et al. (1986) assumes strict convexity of preferences of every consumer, which implies quasi-concavity of the utility function. Proposition 1 shows that, if the normality of the public good is assumed, the existence of Nash equilibrium can be proved even if the utility function is everywhere convex. The proof is based on the observation, initially proposed by Amir and Lambson (2000), that under the condition of normality of public good, every selection of the best-response correspondence $z^*(y_{-i})$ is strictly increasing in y_{-i}, which allows us to use Tarski's fixed-point theorem and to guarantee that every equilibrium is symmetric. Also note that the assumption (A2) for the normality of the private good is not needed here for the existence and symmetry of the Nash Equilibrium.

In addition to the existence and symmetry results, the following Proposition establishes, among other comparative statics results, that the normality of the public good also guarantees that the total maximal and minimal equilibria contributions to the public good are increasing in the number of consumers.

Proposition 2. *With the normality of the public good (A3), the following hold:*

1. The extremal equilibrium joint contributions of $(n-1)$ *consumers to the public good,* $\bar{y}_{-i}^n$ *and* $\underline{y}_{-i}^n$*, are increasing in* n*.*

2. The extremal equilibrium total contribution of all consumers to the public good, $\bar{z}^n$ and $\underline{z}^n$, are increasing in n.

3. The extremal values of indirect utility functions, $\bar{V}_i^n$ and $\underline{V}_i^n$, are increasing in n.

This Proposition, though, does not say anything about how the extremal individual contributions, $\bar{y}_i^n$ and $\underline{y}_i^n$ vary with n. However, the direction of change depends on whether the private good is normal or inferior. Under the normality of the private good as assumed in (A2), we can prove the uniqueness of equilibrium and that the individual contribution $\bar{y}_i^n$ is decreasing in n.

Proposition 3. *With the normality of both the private good (A2) and public good (A3), there exists a unique and symmetric Nash Equilibrium, with per-consumer equilibrium contribution y_i^n decreasing in n and equilibrium consumption of the private good x_i^n increasing in n.*

Proposition 3 presents an alternative proof to the uniqueness result, originally obtained in Bergstrom et al. (1986) and establishes that when both goods are normal, per-consumer maximal and minimal equilibria contributions to the public good are decreasing in the number of consumers. The proof is based on the fact that the normality of both goods implies that every selection of the reaction correspondences $y_i^*(y_{-i})$ has slopes within $[0, -1]$ everywhere, which in turn leads to the uniqueness of equilibrium. On the other hand, the fact that the reaction curves are decreasing (with slopes not greater than 0) is sufficient for the comparative statics results for y_i^n, and $x_i^n = \frac{1}{p}(w - y_i^n)$, given y_{-i}^n increasing in n.

Remark. In a less plausible, but theoretically meaningful case where the private good x_i is inferior, an analogue to Lemma 2 implies that the utility function $U^i(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ will have the strict SCP on $(-y_i; -y_{-i})$, meaning the extremal (all) selections of the best-response correspondence $y^*(y_{-i})$ are increasing by the Monotone Selection Theorem in Milgrom and Shannon (1994). In this case, the uniqueness of equilibrium no longer holds, but it is easily verified that the extremal per-consumer equilibrium contribution $\bar{y}_i^n$ and $\underline{y}_i^n$ will be increasing in n, and the extremal equilibrium consumption of the private good $\bar{x}_i^n$ and $\underline{x}_i^n$ will be decreasing in n, as a direct implication of $\bar{y}_{-i}^n$ and $\underline{y}_{-i}^n$ being increasing in n (Proposition 2). In other words, the comparative statics of the private consumption x_i^n in the group size n depends on whether the private good is normal or inferior.

We conclude this section with a simple example, which shows that the existence result can hold even if the utility function is not quasi-concave, and thus this assumption required by Bergstrom et al. (1986) can indeed be relaxed.

Example 1. Consider the utility function $U^i(x_i, c) = x_i + \sqrt{c}$, which is a strictly quasi-concave function. Now let us assume the production function to be $f(z) = z^2$. By incorporating the production function, the utility function becomes $U^i(x_i, f(z)) = x_i + z$, which is no longer quasi-concave. But Proposition 1 affirms that an equilibrium still exists, and it is symmetric. Indeed, it is easy to check that the maximization problem

$$\max \ (x_i + y_i + y_{-i})$$
$$s.t. \ \ px_i + y_i = w, \ x_i > 0, \ y_i > 0$$

has the following unique solution for any $p > 1$: the dominant strategy of each consumer i, given any y_{-i}, is $x_i^* = 0$ and $y_i^* = w$, $i = 1, ..., n$. So the unique equilibrium is a symmetric one with every consumer contributing w to the public good. Furthermore, we have the total equilibrium contribution $z^n = nw$ and the optimal utility $V_i^n = nw$, which is increasing in n, so the results of Proposition 2 are confirmed.

4 The case with the inferiority of the public good

In some realistic scenarios, the public good may assume some features of an inferior good, such as public parks in certain local areas where it is lower-income families or the homeless that are mostly seen using the parks. In many cities, free public buses or fitness facilities are also overwhelmingly utilized by the financially strained population or tourists. That is, there is reduced consumption of certain public goods when the agents become wealthier.

(A3') *(Inferiority of the public good)* $U_1^i U_{21}^i - U_2^i U_{11}^i < 0.$

In this section, we provide results for the case of an inferior public good. The inferiority is guaranteed by assumption A3', along with an assumption that the consumer is wealthy enough that she consumes and contributes a positive amount of the private good and the public good at her lowest wealth level (when no other consumers contribute to y). The latter guarantees that the solution is interior at the lowest wealth level, circumventing some possible violation to the inferiority (i.e., the consumer will contribute all wealth to the public good without consuming any private good) due to tight budget constraints at low wealth levels, which is by no means a central scenario to our problem. The major implication from the inferiority of the public good is that the utility function $U^i(\frac{1}{p}(w+y_{-i}-z), f(z))$ has the single-crossing property on $(z; -y_{-i})$ (as proved in Lemma 3), implying that the best response $z^*(y_{-i})$ is now decreasing in y_{-i}, which means that the per-consumer contribution $y^*(y_{-i})$ decreases quite rapidly, at slopes less than -1, in other $(n-1)$ consumers' contributions to the public good. Standard lattice-theoretic methods do not guarantee the existence of equilibrium in such submodular games. However, we are able to show that given the normality of the private good (A2),[1] a "single-player provision" equilibrium always exists, where only one consumer contributes to the public good while other consumers do not contribute at all, and this constitutes a mutually best-responding equilibrium. Besides, we show that other equilibria, upon existence, always have the form of m consumers contributing the same amount to the public good while other consumers do not contribute at all, $2 \leq m \leq n$, and no other forms of equilibria besides this can exist. That is, the equilibrium contribution is always symmetric for those active contributors, though non-contributors are allowed to exist.

In addition to the existence arguments, the lattice-theoretic methods also give rise to clear-cut predictions for the comparative statics with respect to the group size n, whenever the equilibrium exists. The major flip of results, by having inferiority of the public good instead of normality, is that the total equilibrium contribution z^n now decreases in n. This means that the free-riding issue compounds with the non-excludability and inferiority of the public good, so that a larger group size may lead to the failure of the public good provision.

Before we state the next Proposition, we need an additional assumption on the utility function to ensure that a consumer has the incentive to donate a positive amount of the public good if no other consumers donate any amount at all. A sufficient condition is that the utility function has a positive partial derivative with respect to y_i at $y_i = 0$ and $y_{-i} = 0$, as stated in the

[1]Since $px_i + y_i = w$, the inferiority of the public implies the normality of the private good. However, the latter does not always guarantee the second-order conditions over U^i and $f(z)$ as stated in A2 to hold, except when x_i is interior where the first-order condition holds. Therefore, we still need assumption A2 for U^i's SCP in $(-y_i; y_{-i})$.

following assumption A4. Hence A4 rules out the possibility of a no-contribution equilibrium and is thus needed for the existence of a single-player provision equilibrium. If otherwise we have $0 \in \arg\max\{U^i(\frac{1}{p}(w-y_i), f(y_i)) : 0 \leq y_i \leq w\}$, then it is easily verified that a trivial equilibrium always exists where no player contributes at all, thus leading to a failure of the public good provision.

(A4) $-\frac{1}{p}U_1^i(\frac{w}{p}, f(0)) + U_2^i(\frac{w}{p}, f(0))f'(0) > 0.$

The next Proposition shows the existence of a single-player provision equilibrium and characterizes other possible equilibria.

Proposition 4. *With the normality of the private good (A2), the inferiority of the public good (A3') and (A4), the following hold:*

1. There always exists an equilibrium in which one consumer contributes a positive amount to the public good and other $(n-1)$ consumers do not contribute at all.

2. Whenever a symmetric equilibrium exists in an m-player game, for some $m < n$, it must be unique, and it also constitutes an equilibrium for the n-player game, with the other $(n-m)$ consumers contributing nothing to the public good.

3. No such equilibrium exists where two active contributors (i.e., $y_i > 0$) have different levels of contribution to the public good.

The second result of Proposition 4 implies that a symmetric m-player equilibrium, upon existence, is invariant to n. That is, if more players join the game, it still constitutes an equilibrium where the same set of m players contributes the same amount, while the new players completely free-ride on existing contributions. The next Proposition gives the comparative statics with respect to n.

Proposition 5. *Under the same assumptions of Proposition 4, whenever the symmetric equilibrium exists for m players and for n players, $n > m$, we have*

$$y_{-i}^n > y_{-i}^m, \quad \textit{and} \quad y_i^n < y_i^m, \ z^n < z^m, \ V_i^n > V_i^m.$$

Next, let us give an example of an inferior public good and a normal private good to show the comparative static results and the existence of a single-player contribution equilibrium.

Example 2. Suppose the production function is $f(z) = z$, and the utility function is $U^i = \ln(f(y_i + y_{-i}) - a) - 2\ln(b - x_i)$, then the maximization problem is

$$\max_{x_i, y_i} U^i = \ln(y_i + y_{-i} - a) - 2\ln(b - x_i)$$
$$s.t.\ \ px_i + y_i = w,\ 0 < x_i < b,\ 0 < y_i < a$$

First, one can verify that $U_1^i U_{21}^i - U_2^i U_{11}^i < 0$, hence y is an inferior good.[2] The assumption A2 can also be verified with ease. Whenever $w \in (\frac{a}{n} + \frac{bp}{2n}, \frac{a}{n} + bp)$, a symmetric equilibrium exists where y^* has an interior solution in $(w - bp, w)$ derived from the first-order condition, which is decreasing in w and y_{-i}, as follows:

$$y^*(y_{-i}) = 2a + bp - w - 2y_{-i}.$$

The symmetric equilibrium is defined by $y^*(y_{-i}) = y_{-i}/(n-1)$, so the equilibrium level of y_{-i} is $y_{-i}^n = \frac{n-1}{2n-1}(2a + bp - w)$, which is increasing in n for $n \geq 2$. The equilibrium individual contribution is $y_i^n = \frac{1}{2n-1}(2a + bp - w)$ and the total contribution is $z^n = \frac{n}{2n-1}(2a + bp - w)$, both decreasing in n, as $\frac{\partial z^n}{\partial n} = \frac{-1}{(2n-1)^2}(2a + bp - w) < 0$. The main takeaway from this example is thus that not only the individual contribution, but also the total contribution to the public good, will decrease when adding more players to the game if the public good is inferior, as opposed to the case of a normal public good. Lastly, notice that if $y_{-i} = 0$, then the best response implies $y^*(0) = 2a + bp - w < w$, so the consumer will contribute $2a + bp - w$ to the public good if no other consumer contributes anything, and for the latter consumer $y^*(2a + bp - w) = 2a + bp - w - 2(2a + bp - w) < 0$, so he will best respond by contributing nothing, and this constitutes a single-player contribution equilibrium, given that the function is well-defined at the points $y_i = 2a + bp - w, y_{-i} = 0$ and $y_i = 0, y_{-i} = 2a + bp - w$.[3]

[2]The logarithm function requires $y_i > w - bp$ for $x_i < b$, so that when y^* coincides with the lowest bound (given it is greater than 0) it increases in w. Nevertheless we do not consider this functional boundary requirement as a violation of the inferiority of y in this specific example.

[3]The function is well-defined for this single-player contribution equilibrium, namely that $0 < y_i = 2a + bp - w < w$ and $w - bp < y_i = 0 < w$ and $y_i + y_{-i} = 2a + bp - w > a$, if $-\frac{bp}{2} < a < 0$ and $a + \frac{bp}{2} < w < 2a + bp$.

5 Conclusion

This paper studies the group size effects in a simple model of private provision of public goods. A major finding of this paper is that the use of the new tools from monotone comparative statics allows us to relax a standard assumption of quasi-concavity of the utility function and to find that the assumption of normality of public good for every consumer and every level of wealth is a sufficient condition for the total contribution to the public good to be increasing with the number of agents. If the public good is instead inferior, combined with the normality of the private good we show that the comparative statics of the total contribution in the group size flips to be decreasing.

References

Amir, R. (1996a). Continuous stochastic games of capital accumulation with convex transitions. *Games and Economic Behavior 15*(2), 111–131.

Amir, R. (1996b). Cournot oligopoly and the theory of supermodular games. *Games and Economic Behavior 15*(2), 132–148.

Amir, R. (2003). Market structure, scale economies and industry performance. *CORE Discussion Paper*.

Amir, R. and V. E. Lambson (2000). On the effects of entry in cournot markets. *The Review of Economic Studies 67*(2), 235–254.

Bergstrom, T., L. Blume, and H. Varian (1986). On the private provision of public goods. *Journal of public economics 29*(1), 25–49.

Cornes, R. and T. Sandler (1984a). Easy riders, joint production, and public goods. *The Economic Journal 94*(375), 580–598.

Cornes, R. and T. Sandler (1984b). The theory of public goods: Non-nash behaviour. *Journal of Public Economics 23*(3), 367–379.

Cornes, R. and T. Sandler (1994). The comparative static properties of the impure public good model. *Journal of public economics 54*(3), 403–421.

Cornes, R. and T. Sandler (1996). *The theory of externalities, public goods, and club goods.* Cambridge University Press.

Gaube, T. (2000). When do distortionary taxes reduce the optimal supply of public goods? *Journal of Public Economics 76*(2), 151–180.

Gaube, T. (2001). Group size and free riding when private and public goods are gross substitutes. *Economics Letters 70*(1), 127–132.

Kerschbamer, R. and C. Puppe (1998). Voluntary contributions when the public good is not necessarily normal. *Journal of Economics 68*, 175–192.

Laffont, J.-J. (1988). *Fundamentals of public economics.* The MIT Press.

McGuire, M. (1974). Group size, group homogeneity, and the aggregate provision of a pure public good under cournot behavior. *Public Choice*, 107–126.

Milgrom, P. and J. Roberts (1990). Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica: Journal of the Econometric Society*, 1255–1277.

Milgrom, P. and J. Roberts (1994). Comparing equilibria. *The American Economic Review*, 441–459.

Milgrom, P. and C. Shannon (1994). Monotone comparative statics. *Econometrica: Journal of the Econometric Society*, 157–180.

Mueller, D. (1989). *Public choice II: A revised edition of public choice.* Cambridge University Press.

Olson, M. L. (1965). *The logic of collective action.* Cambridge, Mass.: Harvard University Press.

Samuelson, P. A. (1954). The pure theory of public expenditure. *The review of economics and statistics 36*(4), 387–389.

Samuelson, P. A. (1955). Diagrammatic exposition of a theory of public expenditure. *The Review of Economics and Statistics 37*(4), 350–356.

Sandler, T. (1992). *Collective action: Theory and applications.* Ann Arbor: University of Michigan Press.

Topkis, D. M. (1978). Minimizing a submodular function on a lattice. *Operations research 26*(2), 305–321.

Topkis, D. M. (1979). Equilibrium points in nonzero-sum n-person submodular games. *Siam Journal on control and optimization 17*(6), 773–787.

Varian, H. R. (1994). Sequential contributions to public goods. *Journal of Public Economics 53*(2), 165–186.

Vives, X. (1990). Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics 19*(3), 305–321.

Warr, P. G. (1982). Pareto optimal redistribution and private charity. *Journal of Public Economics 19*(1), 131–138.

Warr, P. G. (1983). The private provision of a public good is independent of the distribution of income. *Economics letters 13*(2-3), 207–211.

Appendix A1. Proofs

This section provides all the proofs of the paper. First, we introduce some notations that are relevant throughout the proof. A consumer i's best-response correspondence is defined by

$$y^*(y_{-i}; w) = \arg\max\{U^i(\frac{1}{p}(w - y_i), f(y_i + y_{-i})) : 0 \leq y_i \leq w\}, \tag{3}$$

where $y_{-i} \in [0, (n-1)w]$.

Alternatively, one may think of a consumer as choosing the total contribution z of the public good, given $(n-1)$ other consumers' total contribution y_{-i}. So the best-response correspondence in (3) can be rewritten in terms of z as follows:

$$z^*(y_{-i}; w) = \arg\max\{U^i(\frac{1}{p}(w + y_{-i} - z), f(z)) : y_{-i} \leq z \leq y_{-i} + w\}. \tag{4}$$

As we consider homogeneous consumers here, each consumer has the same utility function, so that their best-response correspondences $y^*(y_{-i}; w)$ and $z^*(y_{-i}; w)$ are also the same. Then, as in Amir and Lambson (2000), we introduce the following mapping based on the best-response correspondence (3):

$$\begin{aligned} B_n : [0, (n-1)w] &\to 2^{[0,(n-1)w]}, \\ y_{-i} &\to \frac{n-1}{n}(y_i' + y_{-i}), \end{aligned}$$

where y_i' denotes a best-response level of consumer i's contribution of the public good when the total contribution of other $(n-1)$ consumers is y_{-i}, i.e., $y_i' \in y^*(y_{-i})$. The combined conditions of $y_i \in [0, w]$ and $y_{-i} \in [0, (n-1)w]$ guarantee that B_n maps some y_{-i} into the same space $[0, (n-1)w]$. The mapping B_n is of particular importance while dealing with symmetric equilibria. Indeed, any fixed-point of B_n yields a symmetric Nash equilibrium, as it satisfies $y_{-i} = \frac{n-1}{n}(y_i' + y_{-i})$, or $y_i' = \frac{1}{n-1}y_{-i}$, which means that every consumer donates the same level of the public good.

Next, we establish two important results that connect the normality (inferiority) of the goods to the supermodularity (submodularity) of the game.

Lemma 1. *If A3 holds, then the public good is normal, and the utility function $U^i(\frac{1}{p}(w+y_{-i} - z), f(z))$ has the strict single-crossing property in $(z; y_{-i})$.*

Proof. Inspect the utility functions $U^i(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ and $U^i(\frac{1}{p}(w + y_{-i} - z), f(z))$, noting that the latter is derived by replacing $y_i = z - y_{-i}$. Now we want to show that $U_1^i U_{21}^i - U_2^i U_{11}^i > 0$ implies that U^i has the single-crossing property (SCP) as defined in Milgrom and Shannon (1994) in $(y_i; w)$, and that U^i in the latter form also has the SCP in $(z; y_{-i})$. We prove this by the method of dissection. Let $\tilde{U}(x, y, t) = U^i(\frac{1}{p}(t - x), y)$ and $h(x) = f(x + y_{-i})$. Note that $\tilde{U}$ is completely regular with $\tilde{U}_y = U_2^i > 0$. And $\tilde{U}_x/|\tilde{U}_y| = \frac{-U_1^i}{pU_2^i}$, which is strictly increasing in t if and only if $U_1^i U_{21}^i - U_2^i U_{11}^i > 0$. Therefore, $\tilde{U}$ satisfies the strict Spence-Mirrlees condition and by Theorem 11 in Milgrom and Shannon (1994), $U^i(\frac{1}{p}(t-x), f(x+y_{-i}))$ satisfies the strict SCP in $(x;t)$, whereas $h(x)=f(x+y_{-i})$ can be deemed coming from the richly parameterized family $\{f(x+y_{-i})+\alpha_1 x+\alpha_0 : \alpha_1, \alpha_0 \in \mathbb{R}\}$ with $\alpha_1=\alpha_0=0$. Replacing t by w and x by y_i, we have shown that U^i has the strict SCP in $(y_i;w)$. The procedure to verify that U^i in the latter form has the strict SCP in $(z;y_{-i})$ is similar, if one lets $\tilde{U}(x,y,t)=U^i(\frac{1}{p}(m+t-x),y)$ and $h(x)=f(x)$. Now that constraint set $y_i \in [0,w]$ is ascending in w, and U^i has the strict SCP in $(y_i;w)$, by the Monotone Selection Theorem in Milgrom and Shannon (1994), given any y_{-i}, every selection y^* from the utility maximization (3) is increasing in w, so the normality of y is verified. Q.E.D.

Lemma 2. *If A2 holds, then the private good is normal, and the utility function* $U^i(\frac{1}{p}(w-y_i), f(y_i+y_{-i}))$ *has the strict SCP in* $(-y_i;y_{-i})$.

Proof. We use the method of dissection to prove the normality of the private good and the SCP of U^i. Let $\tilde{U}(x,y,t)=U^i(x,f(t+y))$ and $h(x)=-px+y_{-i}$. Note that $\tilde{U}$ is completely regular with $\tilde{U}_y=U^i_2 f'>0$. And $\tilde{U}_x/|\tilde{U}_y|=\frac{U^i_1}{U^i_2 f'}$, which is strictly increasing in t if and only if $(f')^2(U^i_2U^i_{21}-U^i_1U^i_{22}-\frac{f''}{(f')^2}U^i_1U^i_2)>0$, which is guaranteed by assumption A2. Therefore, $\tilde{U}$ satisfies the strict Spence-Mirrlees condition and by Theorem 11 in Milgrom and Shannon (1994), $U^i(x,f(t-px+y_{-i}))$ satisfies the strict SCP in $(x;t)$, whereas $h(x)=-px+y_{-i}$ can be deemed coming from the richly parameterized family $\{\alpha_1x+\alpha_0 : \alpha_1,\alpha_0\in\mathbb{R}\}$ with $\alpha_1=-p, \alpha_0=y_{-i}$. Replacing t by w, we have shown that U^i has the strict SCP in $(x;w)$. The procedure to verify that $U^i(\frac{1}{p}(w-y_i), f(y_i+y_{-i}))$ has the strict SCP in $(-y_i;y_{-i})$ is similar, if one lets $x=\frac{1}{p}(w-y_i)$, $t=y_{-i}$ and $h(x)=-px+w$. Now that constraint set $x\in[0,\frac{w}{p}]$ is ascending in w, and U^i has the strict SCP in $(x;w)$, by the Monotone Selection Theorem in Milgrom and Shannon (1994), every selection x^* is increasing in w, so the normality of x is verified. Also note that if the opposite inequality holds for A2 and when the public good is normal (A3), it follows that the private good is inferior (the arguments are similar to the latter part of the proof for Lemma 3) and U^i has the strict SCP in $(-y_i;-y_{-i})$. Q.E.D.

Lemma 3. *If A2 and A3' hold, then the public good is inferior, and the utility function* $U^i(\frac{1}{p}(w+y_{-i}-z), f(z))$ *has the strict SCP in* $(z;-y_{-i})$.

Proof. Similar to Lemma 1, $U^i_1U^i_{21}-U^i_2U^i_{11}<0$ implies that U^i has the SCP in $(y_i;-w)$ and that U^i in the latter form has the SCP in $(z; -y_{-i})$, by applying the reverse order on w and y_{-i}. Now we need to show that the public good is inferior, but the Monotone Selection Theorem cannot be directly applied because the constraint set is ascending. We first show that, whenever $y^*(y_{-i}; w)$ is continuous in the interior, i.e., $y^*(y_{-i}; w) < w$, it must be decreasing in w. Suppose not, then there must exist two points on this continuous part of $y^*(w)$ with its corresponding rectangular fully enclosed in the action set, i.e., (w_1, y_1) and (w_2, y_2) such that $w_1 < w_2$, $y_1 < y_2$, and $y_1 \in [0, w_2]$, $y_2 \in [0, w_1]$. Fixing y_{-i}, write $U^i(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ simply as $U^i(y_i, w)$. Then $U^i(y_2, w_2) \geq U^i(y_1, w_2)$, and by the strict SCP, $U^i(y_2, w_1) > U^i(y_1, w_1)$, which contradicts the optimality of y_1 at w_1. So all the continuous interior parts of $y^*(y_{-i}; w)$ are decreasing in w. In addition, $y^*(y_{-i}; w)$ cannot jump up, because then $x^*(y_{-i}; w) = \frac{1}{p}(w - y^*)$ will be jumping down at the same w, contradicting the normality of x (by A2). Since we only consider the case where the consumer will consume and contribute a positive amount of both x and y at the lowest wealth level (see discussions following assumption A3'), it means $y^*(y_{-i}; w)$ will start at an interior point, i.e., less than w, at the lowest possible w, and decrease until it hits the lower boundary 0. Thus the proof is complete for y's inferiority. Q.E.D.

Proof of Proposition 1

Consider the utility maximization problem (4). First, we show that a symmetric equilibrium exists. In Lemma 1, we have proved that under the assumption (A3), we have the normality of public good y_i, and the utility function has strict SCP on $(z; y_{-i})$. In addition, the constraint set $[y_{-i}, y_{-i} + w]$ is also strictly ascending in y_{-i}. By the Monotone Selection Theorem in Milgrom and Shannon (1994), every selection of $z^*(y_{-i})$ is increasing in y_{-i}. Recall that $z^*(y_{-i}) = y_i^*(y_{-i}) + y_{-i}$. Thus, every selection of B_n is increasing in y_{-i}, for any fixed n. By Tarski's fixed-point theorem, B_n has a fixed-point, which is a symmetric Nash equilibrium.

To prove that no asymmetric equilibrium exists, it is sufficient to show that all selections of $z^*(y_{-i})$ are strictly increasing in y_{-i}. Indeed, this would mean that at most one y_{-i} corresponds to each $z' \in z^*(y_{-i})$, s.t. $z' = y_i' + y_{-i}$, with y_i' being the best-response to y_{-i}. But then, for each total contribution z' of public good, each consumer would contribute the same level of public good $y_i' = z' - y_{-i}$, where $y_{-i} = (n-1)y_i'$, implying symmetry in the equilibrium.

Consider an arbitrary selection of $z^*(y_{-i})$, denoted by $\tilde{z}$. To prove that the mapping $y_{-i} \to z^*(y_{-i})$ is strictly increasing, let us assume the contrary: there exist some y_{-i}^1 and y_{-i}^2, with

$y^1_{-i} > y^2_{-i}$, such that $\tilde{z}(y^1_{-i}) = \tilde{z}(y^2_{-i})$, while the equality comes from the fact that every selection of $z^*(y_{-i})$ has been proved to be increasing. Both $\tilde{z}(y^1_{-i})$ and $\tilde{z}(y^2_{-i})$ can be without loss of generality taken to be interior solutions to (4), so each of them satisfies the F.O.C.:

$$U^i_1(\frac{1}{p}(w - z + y^j_{-i}), f(z))(-\frac{1}{p}) + U^i_2(\frac{1}{p}(w - z + y^j_{-i}), f(z))f'(z) = 0, \quad j = 1, 2,$$

where $z \equiv \tilde{z}(y^1_{-i}) = \tilde{z}(y^2_{-i})$. Then the F.O.C. implies:

$$\begin{aligned} &U^i_1(\frac{1}{p}(w - z + y^1_{-i}), f(z))(-\frac{1}{p}) + U^i_2(\frac{1}{p}(w - z + y^1_{-i}), f(z))f'(z) \\ &\qquad = U^i_1(\frac{1}{p}(w - z + y^2_{-i}), f(z))(-\frac{1}{p}) + U^i_2(\frac{1}{p}(w - z + y^2_{-i}), f(z))f'(z), \end{aligned}$$

or

$$\begin{aligned} &-\frac{1}{p}\frac{U^i_1(\frac{1}{p}(w - z + y^1_{-i}), f(z)) - U^i_1(\frac{1}{p}(w - z + y^2_{-i}), f(z))}{\frac{1}{p}(y^1_{-i} - y^2_{-i})} \\ &\qquad + f'(z)\frac{U^i_2(\frac{1}{p}(w - z + y^1_{-i}), f(z)) - U^i_2(\frac{1}{p}(w - z + y^2_{-i}), f(z))}{\frac{1}{p}(y^1_{-i} - y^2_{-i})} = 0. \end{aligned}$$

This holds for all $y_{-i} \in [y^2_{-i}, y^1_{-i}]$. Indeed, as $\tilde{z}$ is increasing, $\tilde{z}(y_{-i}) = z$ for all $y_{-i} \in [y^2_{-i}, y^1_{-i}]$. Hence, we can take a limit as $y^2_{-i} \to y^1_{-i}$ (so $\frac{1}{p}y^2_{-i} \to \frac{1}{p}y^1_{-i}$), and we get

$$-\frac{1}{p}U^i_{11} + U^i_{21}f' = 0 \text{ at } (y^1_{-i}, z). \tag{5}$$

But this is easily seen to violate the assumption (A3), because if we replace $\frac{1}{p} = U^i_2 f'/U^i_1$ due to the F.O.C., $-\frac{1}{p}U^i_1 + U^i_2 f' = 0$, then (5) implies $U^i_2 U^i_{11} - U^i_1 U^i_{21} = 0$, which is a contradiction to (A3). This, in turn, leads us to the conclusion that $\tilde{z}$ is strictly increasing and thus no asymmetric equilibria can exist. Q.E.D.

Proof of Proposition 2

1. Consider the mapping introduced above:

$$B_n : [0,(n-1)w] \to 2^{[0,(n-1)w]},$$
$$y_{-i} \to \frac{n-1}{n}(y_i' + y_{-i}).$$

By the Monotonicity Theorem in Milgrom and Shannon (1994), the facts that the utility function is continuous and has the SCP in $(z; y_{-i})$ and the action set $[y_{-i}, y_{-i}+w]$ is compact and ascending also guarantee that the maximal and minimal selections of $y_{-i} \to z^*$ in (4) exist. It means that the maximal and minimal selections of B_n, denoted by $\bar{B}_n$ and $\underline{B}_n$ respectively, also exist. And it follows from the construction of B_n that the largest equilibrium value of the joint contribution of $n-1$ players, $\bar{y}^n_{-i}$, is also the largest fixed point of $\bar{B}_n$. Since $\frac{n-1}{n}$ is increasing in n, $\bar{B}_n(y_{-i})$ is increasing in n, for every fixed y_{-i}. Then the largest fixed-point of $\bar{B}^n$, which is $\bar{y}^n_{-i}$, is increasing in n due to Milgrom and Roberts (1990). A similar argument applies to establishing that $\underline{y}^n_{-i}$ is increasing in n.

2. Since $\bar{y}^n_{-i}$ is increasing in n and every selection of $y_{-i} \to z^*$ is increasing (from the proof of Proposition 1), the largest total equilibrium contribution to the public good, $\bar{z}^n$, is increasing in n. Similar arguments apply to $\underline{z}^n$.

3. The fact that $\bar{V}^n_i$ is increasing in n follows from the fact that $\bar{y}^n_{-i}$ is increasing in n and the property of the indirect utility function, $V_i(w, y_{-i}) = \max_{0 \le y_i \le w} U^i(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$, to be strictly increasing in other consumers' joint contribution, as

$$\frac{\partial U^i}{\partial y_{-i}} = U^i_2 f' > 0,$$

by the Envelop Theorem. A similar argument applies to $\underline{V}^n_i$. Q.E.D.

Proof of Proposition 3

From Lemma 2, we know that assumption A2 implies that U^i has the strict SCP in $(-y_i; y_{-i})$. Since U^i with a single decision variable is always quasisupermodular, by Milgrom and Shannon (1994) we know any selection of $-y^*(y_{-i})$ is increasing, so any selection of $y^*(y_{-i})$ has slopes bounded above by 0. Recall that in the proof of Proposition 1, we have shown that every selection of $z^*(y_{-i})$ is increasing, where $z^* = y^* + y_{-i}$, which implies that $y^*(y_{-i})$ has slopes bounded below by -1. We can then conclude that all the slopes of every selection of $y^*(y_{-i})$ are in the interval $[-1, 0]$. To prove the uniqueness of Nash equilibrium, we proceed by contradiction and suppose that there exist two Nash equilibria. We note that no asymmetric equilibrium can exist by Proposition 1. Denote by $(y, ..., y)$ and $(y\prime, ..., y\prime)$ two symmetric equilibria, $y \neq y\prime$. Then $(n-1)y \neq (n-1)y\prime$. Suppose $(n-1)y > (n-1)y\prime$, but then $y > y\prime$, and the two points are both Nash Equilibrium thus best responding to y_{-i}, which says that the best-response function is strictly increasing between the two points $((n-1)y\prime, y\prime)$ and $((n-1)y, y)$, which can not be true. This contradiction leads to the conclusion of the uniqueness of the equilibrium. Alternatively, uniqueness also follows from all the slopes of the best-response curves being in the interval $[-1, 0]$ by a standard argument presented and proved in Amir (1996a).

Since we have proved in Proposition 1 that the joint contributions of $(n-1)$ consumers y_{-i}^n are increasing in n, it follows that the per-consumer contribution y_i^n is decreasing in n, due to the downward sloping of $y^*(y_{-i})$. So that $x_i^n = \frac{1}{p}(w - y_i^n)$ is increasing in n. Q.E.D.

Proof of Proposition 4

The following argument regarding the best-response correspondence $y^*(y_{-i})$ is crucial to proving Proposition 4, so we prove this argument first: Under the assumption of normality of the private good (A2) and inferiority of the public good (A3'), every selection $\tilde{y}$ of $y^*(y_{-i})$ from the utility maximization (3) (i) must have slopes no greater than -1 along any continuous part of itself that is interior of $[0, w]$, and (ii) cannot have upward jumps. The reasoning is similar to that of Lemma 3, except that the parameter here is y_{-i} stead of w.

Let us fix w. Under the normality of the private good, we have shown that every such selection $\tilde{y}$ is decreasing in y_{-i} (in the proof of Proposition 3). Specifically, $\tilde{y}$ cannot have upward jumps (but downward jumps are allowed), so the second part (ii) is proved. Since $z^* = y^* + y_{-i}$, it means any selection $\tilde{z}$ of $z^*(y_{-i})$ from the utility maximization (4) cannot have any upward jumps either. Under the inferiority of the public good, Lemma 1 has proved that $U^i(\frac{1}{p}(w+y_{-i}-z), f(z))$ has the strict SCP on $(z; -y_{-i})$. Because the constraint $z \in [y_{-i}, y_{-i}+w]$ is increasing in y_{-i}, so we cannot directly apply the Monotone Selection Theorem. However, for any part of $\tilde{z}$ that is continuously enclosed in the action space, this part of $\tilde{z}$ must be decreasing. This is because if there is a continuous part of $\tilde{z}$ contained in the action space, we can always find two points (y_1, z_1) and (y_2, z_2) on $\tilde{z}$ (potentially close enough), $z_1 = \tilde{z}(y_1)$, $z_2 = \tilde{z}(y_2)$

such that the rectangular defined by this two points are also contained in the action space, i.e., $z_1 \in [y_2, y_2 + w]$ and $z_2 \in [y_1, y_1 + w]$. Without loss of generality let $y_1 < y_2$, then it must be that $z_1 \geq z_2$, because otherwise the fact that z_2 is preferred to z_1 at $y_{-i} = y_2$, would have implied that at $y_{-i} = y_1$, z_2 is (even more) preferred to z_1 due to the (reversely ordered) SCP of U^i, which contradicts the optimality of z_1 at $y_{-i} = y_1$. Then the fact that any selection of z^* must be decreasing along any continuous interior part of itself is equivalent to saying that any selection of y^* must have slopes no greater than -1 along any continuous interior part of itself, so the first part (i) of the argument is also proved. In addition, when $y_{-i} = 0$ the assumption that $y^*(0) < w$ at the lowest wealth level (see discussions following assumption A3') implies that $y^*(0) < w$ for any w, due to its inferiority.

Combining these two facts, we can fully characterize the graph of any selection $\tilde{y}$ from y^*, that $\tilde{y}$ will decrease from the single-player contribution level $y_0 \in (0, w)$ (the strict positivity is guaranteed by A4) at a fast speed with slopes no greater than -1 or with downward jumps, and once it hits 0, it stays on 0 thereafter because no upward jumps are allowed. It implies that any two points in the best-response correspondence cannot have a slope greater than -1 unless one (or both) of the points has hit 0 on the horizontal axis, i.e., with $\tilde{y} = 0$.

1. Now we are ready to prove the existence of of a single-player provision equilibrium where one consumer contributes to the public good and other $(n-1)$ consumers do not contribute at all. Pick some $y_0 > 0$ from the set $y^*(0)$, defined as the individual level of contribution when no other consumer contributes to the public good whose positivity is guaranteed by assumption A4, and it suffices to show $y^*(y_0) = \{0\}$. Suppose not, that there is some $y_1 \in y^*(y_0)$ such that $y_1 > 0$, then we have two points in the best-response correspondence $(0, y_0)$ and (y_0, y_1) such that

$$\frac{y_1 - y_0}{y_0 - 0} = \frac{y_1}{y_0} - 1 > -1,$$

which is contradictory to the implication drawn just above. Therefore, a single-player provision equilibrium always exists with a positive amount of contribution.

2. The existence of a symmetric equilibrium in the n-player game is, however, not guaranteed, and such a failure for the existence of fixed points is prototypical for submodular games. Indeed, as a selection from $y^*(y_{-i})$ can have downward jumps, it may not intersect

$y_{-i}/(n-1)$ for a fixed n, whereas the intersection defines a symmetric equilibrium. However, upon the existence of such a symmetric equilibrium, it must be unique, in view of the strict decreasing $y^*(y_{-i})$ and strictly increasing $y_{-i}/(n-1)$ which can intersect at most once. Now assume such a symmetric equilibrium exists for some $m < n$ wherein each consumer contributes $y_m > 0$ to the public good. Then for the n-player game, m consumers each contributing y_m whereas $(n-m)$ consumers each contributing 0 constitutes an equilibrium. It suffices to show $y^*(my_m) = \{0\}$. The proof is similar to the single-player provision case. Suppose not, that there is some $y_2 \in y^*(my_m)$ such that $y_2 > 0$, then we have two points in the best-response correspondence $((m-1)y_m, y_m)$ and (my_m, y_2) such that

$$\frac{y_2 - y_m}{my_m - (m-1)y_m} = \frac{y_2}{y_m} - 1 > -1,$$

which leads to a contradiction.

3. Lastly, we need to prove that in any equilibrium (upon existence), the active contributors (i.e., with $y_i > 0$) must contribute the same amount of public good. We have shown in the proof of Proposition 1 that the strict inequality in A3 implies that any selection of $z^*(y_{-i})$ must be strictly increasing in its interior part. With an analogous argument, one can show that the strict inequality in A3' implies that any selection of $z^*(y_{-i})$ must be strictly decreasing whenever $z^*(y_{-i})$ is above y_{-i} (which coincides with y_{-i} thereafter once it hits y_{-i}). Indeed, if some selection of $z^*(y_{-i})$ is not strictly decreasing, then the first-order conditions of two interior points, selected to have the same z value, would imply $\frac{\partial^2 U^i}{\partial y_i \partial w} = 0$ at one of the points, thus contradicting the strict SCP. Then if any selection of $z^*(y_{-i})$ is strictly decreasing, one cannot have two active contributors donating different amounts of public good in the equilibrium, because then, suppose consumer i donates more than consumer j, $y_i > y_j > 0$, then we have two points in the best-response correspondence, $(z - y_i, z)$ and $(z - y_j, z)$, with $z \geq y_i + y_j$ and $z - y_j > z - y_i > 0$, contradicting the strict decreasing property of $z^*(y_{-i})$. Q.E.D.

Proof of Proposition 5

Recall that a symmetric equilibrium is defined as the intersection of $y^*(y_{-i})$ and $y_{-i}/(n-1)$. We have characterized the graph of any selection of $y^*(y_{-i})$ to decrease either continuously with slopes no greater than -1 or discontinuously with downward jumps until it hits 0 and then stays at 0 thereafter. Note that $y^*(y_{-i})$ is not affected by n. On the other hand, $y_{-i}/(n-1)$ is a strictly increasing (linear) function of y_{-i}, which rotates clockwise (downwards) when n increases. Therefore, when the number of players increases from m to n, given that the $y^*(y_{-i})$ intersects with both $y_{-i}/(m-1)$ and $y_{-i}/(n-1)$, it is easily seen that the intersection's horizontal coordinate, y_{-i}, increases (note that the intersection cannot happen on the horizontal axis because no-contribution equilibrium is ruled out by assumption A4). Hence $y^n_{-i} > y^m_{-i}$.

As any selection of $y^*(y_{-i})$ decreases strictly in y_{-i} whenever above 0, we have $y_i^n < y_i^m$. For the total contribution z^*, in the proof of Proposition 4(3), we have pointed out that any selection of $z^*(y_{-i})$ must be strictly decreasing whenever $z^*(y_{-i})$ is above y_{-i} (i.e., when y^* is above 0). Therefore, we have $z^n < z^m$. The last inequality $V_i^n > V_i^m$ follows from the fact that the utility function defining V_i strictly increases in other consumers' joint contribution (see proof of Proposition 2(3)), so by Envelop Theorem, $y^n_{-i} > y^m_{-i}$ implies $V_i^n > V_i^m$.

Appendix A2. Lattice-Theoretic Results

Here we present the lattice-theoretic notions and results we used in our analysis. For a thorough discussion of lattice-theoretic methodology the reader is referred to Vives (1990), Topkis (1978, 1979), Milgrom and Roberts (1994).

Let $\geq$ be a binary relation on a nonempty set S. The pair $(S, \geq)$ is a partially ordered set if $\geq$ is reflexive, transitive and antisymmetric. A partially ordered set $(S, \geq)$ is a lattice if any two elements x and y from S have a least upper bound (supremum), $\sup_S(x,y) = \inf\{z \in S : z \geq x, z \leq y\}$, and a greatest lower bound (infimum), $\inf_S(x,y) = \sup\{z \in S : z \leq x, z \geq y\}$.

A lattice $(S, \geq)$ is complete if every nonempty subset of S has a supremum and infimum on S.

A function $g : X \to R$ on the lattice X is supermodular (submodular) if for all x, y in X,

$$g(\inf(x,y)) + g(\sup(x,y)) \geq (\leq) g(x) + g(y). \tag{1.1}$$

The strict supermodularity (submodularity) is defined by a strict inequality in (1.1). For smooth function supermodularity (submodularity) equivalent to the condition $\frac{\partial^2 g(x)}{\partial x_i \partial x_j} \geq (\leq) 0$, for all x, y, the results that is known as Topkis's (1978) Characterization Theorem. Furthermore,

$\frac{\partial^2 g(x)}{\partial x_i \partial x_j} > (<)0$ implies that g is strictly supermodular (submodular).

Let $\phi(t)$ be the interval $[\phi_1(t), \phi_2(t)]$ of reals. $\phi(t)$ is ascending [descending] in t if $\phi_1(t)$ and $\phi_2(t)$ are increasing [decreasing] in t. The following theorem states the monotonicity results on optimal solutions.

Theorem A.1 (Topkis (1978)) *Let $g : X \times T \to R$ be supermodular (submodular) on the lattice X for each t in the partially ordered set T, and $\phi(\cdot)$ is ascending (descending). Then the maximal and minimal selections of $x^*(t) = \arg\max_{x\in\phi(t)}\{g(x,t)\}$ are increasing (decreasing) in t. If g is strongly supermodular (submodular), then every selection of $x^*(\cdot)$ is increasing (decreasing).*

This theorem can be extended to purely ordinal complementarity property called "single crossing property" (SCP).

A function g has a single-crossing property [dual SCP] in (x,t) if for $x' > x$ and $t' > t$,

$$g(x', t) \geq [\leq] g(x, t) \text{ implies that } g(x', t') \geq [\leq] g(x, t')$$

The following theorem provides necessary and sufficient conditions for the monotonicity results.

Theorem A.2 (Milgrom-Shannon (1990)) *If $g : X \times T \to R$ satisfies the SCP (DSCP), and $\phi(\cdot)$ is ascending (descending), then the maximal and minimal selections of $x^*(t) = \arg\max_{x\in\phi(t)}\{g(x,t)\}$ are increasing (decreasing) in t. If g has strict SCP(DSCP), then every selection of $x^*(\cdot)$ is increasing (decreasing).*

The lattice-theoretical fixed-point theorem is due to Tarski (1995).

Theorem A.3. *Let $(S, \geq)$ be a complete lattice, $f : S \to S$ an increasing function, and E the set of fixed points of f. Then E is nonempty and E is a complete lattice.*

The following result is due to Milgrom and Roberts (1990, 1994).

Theorem A.4. *Let $(S, \geq)$ be a complete lattice, $B_t : S \to S$ is a continuous but for upwards jumps function* in x, *for $t \geq 0$ such that $B_t(x)$ is increasing in t, for all x. Then the minimal and maximal fixed points of B_t are increasing in t.*

A continuously differentiable function $U(x, y, t)$ on a rectangular domain with $U_y \neq 0$ satisfies the (strict) Spence-Mirrlees condition if $\frac{U_x}{|U_y|}$ is increasing (increasing) in t for any fixed (x, y).

A family of functions $\{h(\cdot;\alpha) : R \longrightarrow R\}$ is richly parameterized if for all (x', y') and (x'', y'') with $x' \neq x''$, there is some $\hat{\alpha}$ s.t.$y' = h(x', \hat{\alpha})$ and $y'' = h(x'', \hat{\alpha})$.

We close with the following results by Milgrom and Shannon (1994)

Theorem A.5. *Let R^2 be given the lexicographic order. Suppose that $U(x, y, t) : R^3 \to R$ is twice-continuously differentiable and both U_x* and *U_y are nonzero everywhere. Then $U(x, y, t)$ has the (strict) SCP in $(x, y; t)$ if and only if it satisfies the (strict) Spence-Mirrlees condition.*

Theorem A.6. *Suppose that $U(x, y, t) : R^3 \to R$ is twice-continuously differentiable and both U_x* and *U_y are nonzero everywhere. Let $\{h(\cdot;\alpha) : R \longrightarrow R\}$ be a richly parameterized family. Then $U(x, y, t)$ has the (strict) SCP in $(x, y; t)$ if and only if for all $\alpha, g(x; t, \alpha) = U(x, h(x; \alpha), t)$ has the (strict) SCP in (x, t).*