Comparative Statics for the Private Provision of Normal and Inferior Public Good

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Abstract

This paper studies a standard model of the private provision of public goods which can be either normal or inferior. Using new tools from monotone comparative statics, the paper fully characterizes the Nash equilibria for each case and shows that the condition for the normality (inferiority) of the public good is sufficient for the extremal total equilibrium contribution to be increasing (decreasing) in the group size. When the public good is inferior, there always exists a "monopoly provision" equilibrium involving one contributor and $n-1$ free riders, which surprisingly supplies the highest amount of public good among all possible equilibria, also generating the highest social welfare if the utility function is convex in the private good. The lattice-theoretic methodology allows a generalization of the classical results by showing that the assumption of quasi-concavity of the utility function is not "critical" and therefore can be relaxed.

Keywords: Private provision of public goods, monotone comparative statics, single crossing property, group size, free riding

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1 Introduction

The theory of voluntary provision of public goods has received growing attention after the publication of Samuelson's (1954; 1955) seminal works. Since then the theory has been discussed in Olson (1965), McGuire (1974), Laffont (1988), Warr (1982, 1983), Bergstrom et al. (1986), Varian (1994), Kerschbamer and Puppe (1998), Gaube (2000, 2001) and others. One of the most important results was obtained in Bergstrom et al. (1986), which generalized Warr's (1983) invariance result, i.e., the equilibrium contribution to the public good is invariant to moderate income redistribution among players, and gave a full characterization of the equilibrium's comparative statics property with respect to income redistribution under the assumption that both the private good and the public good are normal.

Important contributions to the theory have also been made by Cornes and Sandler (1984a,b, 1994, 1996). They noticed that Nash equilibrium for the provision of a pure public good is generally not efficient, in the sense that the equilibrium contribution to the public good is less than the one that is Pareto efficient. Since then much attention has been engaged to the search for possible solutions that would overcome the under-provision of public good, which is considered as a sign of market failure and a justification for government intervention.

Many works have also been concerned with the problem of free-riding (or easy riding, as Cornes and Sandler call it, since the individuals contribute less than socially desired rather than not contribute at all) and the exacerbation of this tendency as the group size increases. It is a widely accepted hypothesis that when the public good is voluntarily provided, the incentives to free ride and the inefficiency of Nash equilibrium increase with the number of individuals (Olson, 1965; Laffont, 1988; Mueller, 1989; Sandler, 1992). But this claim has been usually illustrated only by means of examples of Cobb-Douglas and quasilinear utility functions. Gaube (2001) found that sufficient conditions for this general presumption to hold are conditions for both public and private goods to be strictly normal and weak gross substitutes.

Another important issue in the theory of public goods is whether a sequential model of private provision leads to greater or lesser contribution than a simultaneous-move game. Varian (1994) studies both models and shows that under the strict normality of both the private good and the public good, the total contribution to the public good in a sequential game is never larger than in a simultaneous-move game.

Overall, the assumption of strict normality of both goods for all consumers is easily seen to be standard in the theory of private provision of public goods. The purpose of the present paper is to see whether this assumption, widely used in the theory, is sufficient to give rise to unambiguous comparative statics conclusions without reliance on other, nonessential assumptions (such as the strict quasi-concavity of the utility function), and how the reversal of this assumption to inferior public goods, affects the overall comparative statics analyses. As pointed out by Kerschbamer and Puppe (1998), the strict normality of public goods at any income level is not well justified. The notion of inferior public goods is not a far-fetched one on practical grounds: The public parks and buses in many metropolitan areas are predominantly utilized by the lower-income groups; people tend to substitute private services for public facilities (such as private gym for community recreaction center) when they become wealthier; etc.

Building on the classical model studied in Warr (1983); Bergstrom et al. (1986), this paper studies the two cases separately (i.e., when the public good is normal and when it is inferior), fully characterizes the pure-strategy Nash equilibria, and gives a thorough examination of the equilibrium's comparative statics property with respect to the group size.

The analysis of this paper relies on a new monotone comparative statics approach based on lattice-theoretic methods (Topkis, 1978, 1979; Vives, 1990; Milgrom and Roberts, 1990, 1994; Milgrom and Shannon, 1994). The advantage of this new approach over the traditional method is widely recognized as utilizing only a subset of the standard assumptions to deliver general, unambiguous comparative statics results. In other words, it discards the superfluous assumptions that are not "critical" to deriving the results, despite their wide usage in the theory literature. Besides, this approach also circumvents some ill-defined comparative-statics statements arising from the multiplicity of equilibria, by dealing only with the extremal equilibria. One major application of the monotone comparative statics approach is in oligopoly theory (see, among others, Amir, 1996b, 2003; Vives, 1999; Amir and Lambson, 2000).

The standard model used in the theory literature consists of one public good, one private good, and n consumers with identical tastes. Unlike Bergstrom et al. (1986), this paper also assumes all consumers have the same wealth thus focusing on the symmetric equilibrium. Adding a production technology for the public good, the utility function may no longer possess the property of strict quasi-concavity (in the decision variables) that is generally assumed to be true in the classical literature. Nevertheless, the lattice-theoretic method confirms the existence of a symmetric Nash equilibrium as long as the public good is normal, by using Tarski's fixed-point theorem for isotone (increasing) reaction mappings. In this case, there may be multiple symmetric equilibria but no other asymmetric equilibrium exists, and the total public good supply in the extremal equilibria is increasing in n. If the normality of the private good is also imposed, then the equilibrium is unique, and the individual contribution to the public good is decreasing in n. Hence these results may suggest a moderate form of free-riding: while each individual contributes less with a larger group size, the total contribution still increases.

The novel case with an inferior public good is studied in the second part of the paper. It turns out that the inferiority of the public good implies strong submodularity between the players' contributions: the reaction curve decreases at a rapid speed with slopes no greater than -1. It is well-known that no general existence of Nash equilibrium is guaranteed in submodular games (with a few exceptions, see e.g., Vives, 1999). Nevertheless, a "monopoly provision" equilibrium is identified and is shown to always exist with an inferior public good. The "monopoly provision" equilibrium refers to one wherein one consumer contributes and everyone else free rides. The existence is due to the afore-mentioned strongly decreasing reaction mappings, an observation first made by Novshek (1985) and applied to Cournot context by Amir and Lambson (2000).

In addition to the 'monopoly provision" equilibrium, other possible equilibria are also characterized, which must be partially symmetric (in the sense that all contributors must donate the same amount). Comparing the symmetric equilibria, the inferiority of the public good leads to a reversal in the comparative staitcs: not just the individual contribution, but the equilibrium total contribution also decreases in n. In other words, a larger group will end up in providing less public good, a quite counter-intuitive result marking a strong form of free-riding. A direct implication is that fixing the group size, the "monopoly provision" equilibrium provides the highest amount of public good among all possible equilibria! Furthermore, if the utility function is convex in the private good (which coincides with the condition for the inferiority of the public good if the utility function is separable, also see Liebhafsky, 1969), the "monopoly provision" equilibrium also generates the highest social welfare among all possible equilibria.

The rest of the paper is organized as follows. The general model is developed in Section 2, with some discussions on the premises of the model. The case with a normal public good is analyzed in Section 3, followed by the case with an inferior public good in Section 4. Section 5 concludes. All proofs of the theorems are placed in the Appendix.

2 The model and fundamental premises

This section lays out the fundamentals of a simple model of private provision of public good, which is a variation of the earlier version discussed in Bergstrom et al. (1986). A major extension of the paper over the classical models is to introduce a production technology for the public good, taking the voluntary contribution of players as the sole input. The introduction of this production technology may undermine the regularity assumption for the utility function, which requires it to be strictly quasi-concave (in the decision variables).

2.1 The model setup

Consider a standard utility maximization with one public good, one (representative) private good, and n consumers (players) in a community. Each consumer has the same wealth w and preference $U(x_i, q)$. Here, x_i is the amount of private good purchased by consumer i, and q is the consumption/output of the public good, which is the same for everyone. The public good is produced using a sole input, which is voluntarily donated by the consumers. Let y_i denote the contribution of consumer i, y_{-i} the sum of the contribution made by all other consumers except consumer i. The total contribution made by all consumers is denoted by z, i.e., $z = \sum_{i=1}^{n} y_i$ or $z = y_i + y_{-i}$ for any i. The production technology is specified by a strictly increasing production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ which maps the input z to the output q of the public good, i.e., $q = f(z)$. Even without contributing from herself, consumer i enjoys a total consumption $q = f(y_{-i})$ of the public good, which creates an incentive to free ride. In the classical literature (e.g., Warr, 1982; Bergstrom et al., 1986; Cornes and Sandler, 1984a,b; Varian, 1994), f usually assumes the degenerate form of the identity function, i.e., $f(z) = z$, in the sense that the donations made by the players can be consumed directly (e.g., books) or there is constant return to scale.

The prices of the two marketed goods are exogenous: the private good is traded at a price p, and the price of the public good input is without loss of generality normalized to 1. Thus one may think of the contribution as a pure monetary one. Taking the voluntary contribution by other consumers y_{-i} as given, consumer i maximizes her utility $U(x_i, f(y_i + y_{-i}))$ by choosing her private consumption x_i and the voluntary contribution y_i, subject to the usual budget constraint $px_i + y_i = w$.

Definition 1. *A Nash Equilibrium for the model of private provision of the public good is a vector of contributions $y_i^*, i = 1, ..., n$, such that for each i, (x_i^*, y_i^*) is a solution to the following utility maximization problem*

$$\begin{aligned} &\max_{x_i \geq 0, y_i \geq 0} U(x_i, q) \\ s.t. \quad & px_i + y_i = w \\ & q = f(y_i + y_{-i}^*) \end{aligned}$$

The major interest of the paper lies in studying the comparative statics of the equilibrium public good contribution (y_i for each individual and z for the community) with respect to the group size n. One may notice the resemblance of the underlying structure to that of a Cournot oligopoly, where the conduct of comparing the market performance variables such as the firm's and the industry output in a Cournot-Nash equilibrium under different market structures is often seen (a more explicit connection is studied in McGuire, 1974). Indeed, the key technique used in this paper, namely by examining the players' reaction correspondences under symmetric assumptions in a supermodular (submodular) game, bears a close resemblance to that used by Amir and Lambson (2000) in studying the comparative statics in a Cournot oligopoly.

As for the notation, an upper script n is used to denote the equilibrium set (or singleton if the equilibrium is unique) value of the variables in a n-player symmetric equilibrium. That is, x_i^n denotes the equilibrium private good consumption (the lower script i is kept to indicate the variable evaluated at an individual level), y_i^n the equilibrium individual contribution to the public good, z^n the equilibrium total contribution, etc. If there is a possibility of multiple equilibria, i.e., the equilibrium variables have a set value, the upper and lower bars are used to denote the maximal and minimal elements of the equilibrium set of variables, i.e., the maximal and minimal equilibrium. With a simple application of the Envelop Theorem, one may also derive the comparative statics for the consumer's equilibrium utility value. The indirect utility function of consumer i is denoted by $V(w, y_{-i})$, which is a function of the wealth w and the contribution made by other players y_{-i}. Formally, substituting x_i from the budget constraint, $V(w, y_{-i})$ is defined as

$$V(w, y_{-i}) = \max_{0 \leq y_i \leq w} U(\frac{1}{p}(w - y_i), f(y_i + y_{-i})). \tag{1}$$

As noted in Warr (1982) and Bergstrom et al. (1986), an alternative way to formulate the problem is to view consumer i as choosing the total contribution level $z = y_{-i} + y_i$, given the others' contribution y_{-i}. The feasibility constraint becomes $y_{-i} \leq z \leq y_{-i} + w$, where the right-hand side can be viewed as the consumer's adjusted wealth, which equals her own wealth plus the contribution made by others. Such a change of variable is widely seen in the context of aggregate games, where the player's payoff depends on the aggregate value of other players' actions, the public good game at hand being one of such a feature while another being the Cournot oligopoly game (see, e.g., Novshek, 1985; Amir and Lambson, 2000). Thus, the corresponding utility maximization after substituting x_i from the budget constraint becomes

$$\max_{y_{-i} \leq z \leq w + y_{-i}} U(\frac{1}{p}(w + y_{-i} - z), f(z)). \tag{2}$$

The two alternative specifications of utility maximization (1) and (2) should yield consistent solutions to the consumer problem. A joint examination of both is later shown to be essential in establishing the properties of the players' reaction correspondences, which facilitates the comparative-statics analysis.

2.2 The premises and the normality of the private good

This subsection gives a detailed account of the premises of the model based on the primitives. For regularity, the following assumptions are valid throughout the paper:

(A0) *$f(z)$ is strictly increasing and twice continuously differentiable.*

(A1) *$U(x_i, q)$ is twice continuously differentiable with $U_1, U_2 > 0$.*

(A2) *(Normality of the private good) $U_2 U_{21} - U_1 U_{22} - \frac{f''}{(f')^2} U_1 U_2 > 0$.*

The production function is monotone. The utility function is strictly increasing in both arguments in a global sense, reflecting the view that both of the private and public commodities are good without satiation (an interesting discussion for the Pareto properties of a satiating utility function can be found in Cornes and Sandler, 1984a). Note that the smoothness is assumed for ease of exploration but is not essential to the analysis.

In particular, there is no explicit restriction on the utility function being quasi-concave, which is usually taken as a standard assumption in the classical literature validating the use of the Implicit Function Theorem or the standard fixed-point theorems. The production function adds another layer of intricacy. Even with a well-behaved, convex preference, so that $U(x_i, q)$ is strictly quasi-concave in the two goods (x_i, q), $U(x_i, f(z))$ may fail to be strictly quasi-concave in the two decision variables (x_i, z), in light of a rather convex production technology $f(\cdot)$ when there is strong decreasing return to scale (this point is illustrated in Example 1). Indeed, a well-known advantage of the lattice-theoretic methodology used in this paper is to discard superfluous assumptions as such and to yield general, unambiguous comparative-statics conclusions with a minimally sufficient set of conditions.

A2 is a sufficient condition for the private good to be normal. The normality is defined in the usual sense, i.e., given any y_{-i}, (every selection of) the arg max of the utility maximization,

$$x^*(y_{-i}; w) = \arg\max\{U(x_i, f(w - px_i + y_{-i})) : 0 \leq x_i \leq \frac{w}{p}\} \tag{3}$$

is an increasing function of w. Notice that if $f(z) = z$, A2 becomes the standard normality condition, $U_2U_{21} - U_1U_{22} > 0$, or equivalently the marginal rate of substitution $U_1(x_i, q)/U_2(x_i, q)$ increasing in q. As is well known, the normality of the private good implies that the player's optimal contribution along the reaction curve path decreases in the sum of other players' contributions.[1] A2 is less (more) restrictive than the standard normality condition if the production function has decreasing (increasing) return to scale, i.e., $f'' < 0$ ($f'' > 0$).

Because the utility function need not necessarily be quasi-concave, the sufficiency of A2 in

[1]A simple way to see this is to assume consumer i's demand of the public good (input), z, given her adjusted wealth $w_i + y_{-i}$, is $z = g(w_i + y_{-i})$. Subtracting y_{-i} from both sides, then her willingness to contribute is $y_i = g(w_i + y_{-i}) - y_{-i}$. Obviously, $\frac{dy_i}{dw_i} = g'$, and $\frac{dy_i}{dy_{-i}} = g' - 1$, the latter being the slope of player i's reaction curve. Since $\frac{dy_i}{dw_i} + p\frac{dx_i}{dw_i} = 1$, a normal private good implies $\frac{dy_i}{dw_i} \leq 1$, or $\frac{dy_i}{dy_{-i}} \leq 0$. Similarly, a normal (inferior) public good implies $\frac{dy_i}{dw_i} \geq (\leq) 0$, or $\frac{dy_i}{dy_{-i}} \geq (\leq) -1$. This simple illustration apparently requires the existence of a continuous demand function of the public good, $g(\cdot)$, which in turn requires strict quasi-concavity of the utility function. This is not needed in the formal analysis using the lattice-theoretic arguments.

implying the normality of the private good as well as the negative monotonicity of the players' reaction correspondences cannot be directly shown by the conventional method, which relies on using the Implicit Function Theorem. Instead, a lattice-theoretic argument is established here that entails showing a two-fold implication of A2: (i) the utility function $U(x_i, f(w-px_i+y_{-i}))$ as specified in maximization (3), has the strict single crossing property or SCP (Milgrom and Shannon, 1994) in $(x_i; w)$, and similarly, (ii) $U(\frac{1}{p}(w-y_i), f(y_i+y_{-i}))$ as specified in maximization (1), has the strict SCP in $(-y_i; y_{-i})$, i.e., with a reversed order on y_i.

Definition 2 (Milgrom and Shannon, 1994)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$. Then f satisfies the (strict) single crossing property in $(x; t)$ if for $x' > x''$ and $t' > t''$, $f(x', t'') > f(x'', t'')$ implies that $f(x', t') \geq (>) f(x'', t')$.*[2]

Being the ordinal version of the property of increasing differences for a function, the SCP similarly describes the complementarity between a function's two arguments, but with a discrimination between the two variables (in the sense that x and t are not interchangeable). The direct application of the SCP is to establish comparative statics with respect to the two arguments considered at hand. The related theorem is proposed in Milgrom and Shannon (1994) as the Monotonicity Selection Theorem, on which all of this paper's results are based. The Monotonicity Selection Theorem is an important ordinal generalization of Topkis's (1978) result on supermodular functions (or functions with increasing differences in the Euclidean space), which is essentially a cardinal property and hence need not be preserved with monotone transformations. In short, the theorem says that every selection of the arg max, $x^*(t) = \arg\max\{f(x; t) : x \in [h(t), g(t)]\}$, is increasing in t if $f(x; t)$ has the strict SCP in $(x; t)$ and both $h(t)$ and $g(t)$ are nondecreasing in t.

In the public good game context, the Monotonicity Selection Theorem suggests that (with a qualification of the feasibility constraint that can be easily checked here), as implied by A2, since $U(x_i, f(w-px_i+y_{-i}))$ has the SCP in $(x_i; w)$, the utility maximization with respect to x_i necessarily yields upward sloping correspondence $x^*(y_{-i}; w)$ with respect to w, which implies the

[2]The single crossing property is the ordinal version of increasing differences (or supermodularity in the Euclidean space at hand), which is preserved under monotonone transformation of the objective function. SCP discriminates between the two variables, and is more general than increasing differences in that a function satisfying the latter necessarily satisfies the former, e.g., the condition for increasing differences of a continuously differentiable function $\partial^2 f/\partial x \partial t \geq 0$ implies that f has the SCP in $(x; t)$ as well as in $(t; x)$.

normality of x_i. Similarly, since $U(\frac{1}{p}(w-y_i), f(y_i+y_{-i}))$ has the SCP in $(-y_i; y_{-i})$, the utility maximization with respect to y_i necessarily yields downward sloping correspondence $y^*(y_{-i}; w)$ with respect to y_{-i}, i.e., downward sloping reaction curves. The significance of A2 is formally addressed in Lemma 1.

Lemma 1. *If A2 holds, then (i) the private good is normal, and (ii) every selection of the reaction correspondence,* $y^*(y_{-i}; w) = \arg\max\{U(\frac{1}{p}(w-y_i), f(y_i+y_{-i})) : 0 \leq y_i \leq w\}$*, is decreasing in* y_{-i} *for any* w*.*

Under A2, the private good is assumed to be normal throughout the paper—with only one representative private good in the consumer's utility function, one can hardly suppose the opposite—though a remark on some reversed results of the comparative statics of the individual equilibrium contribution, when the private good is inferior while the public good is normal, will be given in the next section. As one may conjecture, the private good is inferior if the inequality in A2 holds in the "less than" direction,[3] which also implies that the individual equilibrium contribution is an increasing function of the other players' contribution, a case where there is no free riding! In fact, this situation exactly corresponds to the "perverse" case in Cournot oligopoly where a firm's reaction curve is increasing in the rival's output (see e.g., Amir, 1996b; Vives, 1999), which yields unintuitive conclusions for the market performance variables.

In the following paper, two split cases on the characteristics of the public good are discussed in order: (1) when the public good is normal and (2) when it is inferior. Another assumption similar to A2 will be shown to guarantee the normality or inferiority of the public good. Before proceeding to the discussion of public goods, some last comments on the significance of A2 are needed in place. First, in the case when the public good is normal, A2 is not needed for the existence of a pure-strategy Nash equilibrium (Proposition 1) or the comparative statics of the equilibrium total contribution z^n and the indirect utility function V^n with respect to n (Proposition 2), but it is crucial in determining the comparative statics of the equilibrium individual contribution y_i^n, to be decreasing in n (Proposition 3). A reversion of A2 leads to the opposite, atypical if not perverse result. If a player is deemed to have a free-riding incentive

[3] Because the feasibility constraint, $x_i \in [0, w/p]$, is ascending in w, some additional arguments are needed to ascertain the inferiority of x_i, namely the private good needs to be normal in the premise. A detailed remark is given in the Appendix following the proof of Lemma 1.

when her reaction curve is decreasing in other players' contribution, then one may say that the normality of the private good is what drives the free-riding incentive. On the other hand, A2 must hold in the second case, when the public good is inferior, as a natural consequence of the two-good economy considered at hand.

Remark. The results of the paper also apply to a setting with n firms maximizing their output of a product (or their profits if the product market is perfectly competitive), where the production function of Firm i depends on a private input x_i (good 1) and a collective input z (good 2), with $z = y_i + y_{-i}$, so that each firm uses not only its own input y_i of good 2 but also what is available from others' input (such as R&D investments with perfect spillovers).

3 The case with a normal public good

In this section, I consider the case when the public good is normal for each consumer i, $i = 1, ..., n$. The public good is said to be normal if given any y_{-i}, every selection of the arg max of (1), $y^*(y_{-i}; w)$ is increasing in w. In other words, the individual contribution given other players' contribution is an increasing function of her wealth. This is consistent with the alternative definition of normality over the output or consumption of a good, as the monotonicity is preserved by the strictly increasing production function (A0), with $q = f(y^*(y_{-i}; w) + y_{-i})$.

In Bergstrom et al. (1986), the normality of both the private and public goods is shown to guarantee a unique pure-strategy Nash equilibrium, giving rise to the comparative statics that pertain to the redistribution of wealth. Here, the focus is instead on the comparative statics pertaining to the group size n.

The following assumption is sufficient for the normality of the public good.

(A3) *(Normality of the public good)* $U_1 U_{21} - U_2 U_{11} > 0$.

Lemma 2. *If A3 holds, then (i) the public good is normal, and (ii) every selection of the reaction correspondence,* $z^*(y_{-i}; w) = \arg\max\{U(\frac{1}{p}(w + y_{-i} - z), f(z)) : y_{-i} \leq z \leq w + y_{-i}\}$, *is increasing in* y_{-i} *for any* w.

A3 is the familiar condition for a normal public good. Expressed in an equivalent way, it requires the MRS, $U_2(x_i, q)/U_1(x_i, q)$, to be increasing in x_i. The direct implication of A3 is that $U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ as specified in (1) has the strict SCP in $(y_i; w)$, and that $U(\frac{1}{p}(w + y_{-i} - z), f(z))$ as specified in (2) has the SCP in $(z; y_{-i})$. Then Lemma 2 follows by applying the Monotonicity Selection Theorem.

Therefore, the total contribution chosen by a player is increasing in the other players' contributions. Also, $z^*(y_{-i}; w)$ is allowed to be not everywhere continuous, but can only have upward jumps, as suggested by Lemma 2(ii). Note that for any selection $y' \in y^*(y_{-i}; w)$, there exists some $z' \in z^*(y_{-i}; w)$ such that $z' = y' + y_{-i}$. It follows that any selection of $y^*(y_{-i}; w)$ has its slopes bounded below by -1. The monotone property of $z^*(y_{-i}; w)$ allows us to establish the existence of a symmetric Nash equilibrium by Tarski's fixed-point theorem and to rule out any asymmetric equilibria.

Proposition 1. *With the normality of the public good (A3), there exists a symmetric Nash equilibrium, and no asymmetric equilibria exist.*

A symmetric Nash equilibrium is a fixed point of the selection $\frac{n-1}{n} z'(y_{-i})$, where $z'(y_{-i}) \in z^*(y_{-i}; w)$ with the wealth argument omitted. Because any such equilibrium must satisfy $y'(y_{-i}) = \frac{1}{n-1} y_{-i}$ where $y'(y_{-i})$ is a selection of the reaction correspondence $y^*(y_{-i}; w)$, and then $z'(y_{-i}) = y'(y_{-i}) + y_{-i} = \frac{n}{n-1} y_{-i}$. By Lemma 2, every selection of $z^*(y_{-i}; w)$ is increasing in y_{-i}, then the existence of a symmetric Nash equilibrium follows by Tarski's fixed-point theorem. In contrast to Bergstrom et al. (1986) where the existence is obtained by assuming strictly quasi-concave utility function and applying Brouwer's fixed-point theorem for a continuous function, here the utility function $U(x_i, f(z))$ need not be quasi-concave in (x_i, z). Furthermore, the strict inequality of A3 implies that every selection of $z^*(y_{-i}; w)$ is strictly increasing in y_{-i}, hence there is only one y_{-i} consistent with a player's choice of the equilibrium total contribution z. This assures that no asymmetric equilibrium can exist.

With upward-sloping reaction correspondences, the game may have more than one symmetric equilibrium. The possibility of multiple equilibria leads to criticism of the traditional method used to analyze the comparative statics of equilibrium variables for its ambiguity, namely by differentiating the first-order condition and analyzing the sign of certain partial derivatives. Because not all equilibria satisfy the comparative statics derived this way—in general, only the stable ones do—and the number of equilibria may also change. In this regard, the lattice-theoretic method holds an advantage as it yields general, unambiguous comparative-statics conclusions for the extremal (i.e., the maximal and minimal) equilibria, usually with a minimally sufficient set of complementarity conditions. The next Proposition states the comparative-statics results for several important variables.

Proposition 2. *With the normality of the public good (A3), the following hold:*

1. The joint contributions of $(n-1)$ consumers to the public good in the extremal equilibria, $\bar{y}^n_{-i}$ and $\underline{y}^n_{-i}$, are increasing in n.

2. The total contribution of all consumers to the public good in the extremal equilibria, $\bar{z}^n$ and $\underline{z}^n$, are increasing in n.

3. The value of the indirect utility function in the extremal equilibria, $\bar{V}^n$ and $\underline{V}^n$, are increasing in n.

The proof of this set of comparative-statics results is straightforward. Since the extremal selections of the reaction correspondence $\frac{n-1}{n}z^*(y_{-i})$ are increasing in n (i.e., $\frac{n-1}{n}$ increases in n and $z^*(y_{-i})$ is not affected by n), the extremal fixed points $\bar{y}^n_{-i}$ and $\underline{y}^n_{-i}$ of these selections must also increase in n, as they are the intersection of the reaction functions with the 45-degree line. Then $\bar{z}^n = \bar{z}^*(\bar{y}^n_{-i}; w)$ and $\underline{z}^n = \underline{z}^*(\underline{y}^n_{-i}; w)$ are increasing in n, because $\bar{z}^*(y_{-i}; w)$ and $\underline{z}^*(y_{-i}; w)$ are increasing functions of y_{-i}. The same arguments hold for the indirect utility function too, because $V(w, y_{-i})$ is an increasing function of y_{-i} by the Envelope Theorem. When the public good is normal, the Proposition confirms that the total public supply increases with the group size and each consumer is better off in the extremal equilibria. However, it is unclear whether the individual contribution also increases with n as the total contribution may have risen due to the added number of consumers. In that sense, the extremal individual contribution $\bar{y}^n_i$ and $\underline{y}^n_i$ can be either increasing or decreasing in n. It turns out the comparative statics of individual contribution depends on the characteristic of the private good, which is so far not needed, but crucial to the next Proposition.

Proposition 3. *With the normality of both the private good (A2) and public good (A3), there exists a unique and symmetric Nash Equilibrium, where the equilibrium individual contribution y^n_i decreases in n, and the private good consumption x^n_i increases in n.*

Lemma 1 and 2 imply that the normality of both goods guarantees the slope of every selection of the reaction correspondence $y^*(y_{-i}; w)$ to be bounded between $[-1, 0]$. The uniqueness of Nash equilibrium then follows from a fairly standard argument (see, e.g., Lemma 2.3 in Amir, 1996a), and a simple alternative proof (using the fact that no asymmetric equilibria exist) is also provided in the Appendix. The comparative statics follows, as every selection of $y^*(y_{-i}; w)$ is decreasing in y_{-i}, and the equilibrium y^n_{-i} is increasing in n by Proposition 2. Notice that the maximal and minimal equilibria coincide due to the uniqueness of Nash equilibrium.

Although the conclusions are derived using the lattice-theoretic method, the normality of both the private and public good also implies that the utility function is quasi-concave,[4] so the uniqueness can also be proved by the classical methodology (as derived in Theorem 3 of Bergstrom et al., 1986).

Remark. Another possible case is that the public good is normal, but the private good is inferior. It is less plausible as the utility function consists of only one private good. Nevertheless, this case may still carry some theoretical interest and a brief remark is given here. It is proved in the Appendix (following the proof of Lemma 1) that if A2 holds in the opposite direction, the utility function $U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ has the strict SCP in $(y_i; y_{-i})$, and hence every selection of the reaction correspondence $y^*(y_{-i}; w)$ is increasing in y_{-i} by the Monotonicity Selection Theorem (Milgrom and Shannon, 1994). It means that rather than being an easy rider (as called by Cornes and Sandler, 1984a), a player will increase her contribution along the reaction curve when she observes more contribution from other players; the free-riding incentive disappears if the private good is inferior. The upward sloping reaction correspondence may lead to multiple equilibria. But as Propositions 1 and 2 still hold in this scenario, the contribution of $n-1$ players in the extremal equilibria, $\bar{y}^n_{-i}$ and $\underline{y}^n_{-i}$, are increasing in n, so the equilibrium individual contribution $\bar{y}^n_i = \bar{y}(\bar{y}^n_{-i}; w)$ and $\underline{y}^n_i = \underline{y}(\underline{y}^n_{-i}; w)$ will increase in n, and the equilibrium private good consumption $\bar{x}^n_i$ and $\underline{x}^n_i$ will decrease in n. In a nutshell, conclusions given in Proposition 3 have a major reversion when the private good is inferior.

[4]The utility function $U(x_i, f(z))$ is quasi-concave in (x_i, z) if the determinant of the bordered Hessian is positive, i.e., $2U_1U_2U_{21} - U_2^2U_{11} - U_1^2U_{22} - \frac{f''}{(f')^2}U_1^2U_2 \equiv D > 0$. Note that the determinant can be decomposed in such a way as $D = U_1(U_2U_{21} - U_1U_{22} - \frac{f''}{(f')^2}U_1U_2) + U_2(U_1U_{21} - U_2U_{11})$. Therefore if both A2 (normality of private good) and A3 (normality of public good) hold, the utility function is quasi-concave in (x_i, z). The failure of either of the two assumptions may result in U not being quasi-concave.

This section is concluded with a simple but illustrative example, where the utility function is strictly quasi-concave in (x_i, q)—thus the preference is strict convex with respect to the private the public good—but is not strictly quasi-concave in the two decision variables (x_i, z) after incorporating the production function $q = f(z)$. If the private good is more expensive than the public good input (i.e., $p > 1$), each player has a dominant strategy to contribute all wealth to the public good, i.e., $x^*(y_{-i}; w) = 0$ and $y^*(y_{-i}; w) = w$. Notice that the former is weakly increasing (i.e., constant) in w and the latter is weakly decreasing in y_{-i}, thus this is a borderline example in light of Lemma 1. The comparative-statics conclusions can be checked procedurally.

Example 1. *Consider the utility function $U(x_i, c) = x_i + \sqrt{q}$. The utility represents a well-behaved convex preference and is obviously strictly quasi-concave in (x_i, q). Assume the production function is convex and takes the form $f(z) = z^2$. Incorporating $f(\cdot)$, the utility function becomes $U(x_i, f(z)) = x_i + z$, linear but no longer strictly quasi-concave, which do not fit with the traditional method. Nevertheless, there exists a unique, symmetric Nash equilibrium corresponding to the utility maximization*

$$\max \ (x_i + y_i + y_{-i})$$
$$s.t. \ \ px_i + y_i = w, \ x_i > 0, \ y_i > 0.$$

Note that for any $p > 1$, the dominant strategy for each player i is to choose $y^*(y_{-i}; w) = w$ and $x^*(y_{-i}; w) = 0$ as the marginal utility is 1 for both goods, no matter what y_{-i} the opponents choose. Then $z^*(y_{-i}; w) = w + y_{-i}$, which is increasing in y_{-i}. The unique dominant-solvable equilibrium is a symmetric one where every consumer contributes all her wealth to the public good. Then the equilibrium contribution of $n-1$ players is $y^n_{-i} = (n-1)w$, which is increasing in n. In the equilibrium, the total contribution is $z^n = nw$ while the indirect utility of each player is $V^n = nw$, both are increasing in n. The equilibrium individual contribution, $y^n_i = w$, is weakly decreasing in n. If $p < 1$, there is also a unique, dominant-solvable equilibrium where no one contributes at all, i.e., $x^n_i = w/p$, $y^n_i = 0$, $y^n_{-i} = z^n = 0$ and $V^n = w/p$. As seen, all the comparative statics results hold in a borderline sense.

4 The case with an inferior public good

In the classical literature on the private provision of public goods, a standard assumption is the strict normality of both the private good and the public good. This assumption seems to be mainly motivated by technical reasons, as the dual normality and the strict quasi-concavity of the utility function jointly guarantee the existence of a single-valued demand of the public good as a function of the individual wealth, with its slopes bounded between 0 and 1 (see e.g., Bergstrom et al., 1986). However, as pointed out by Kerschbamer and Puppe (1998), there are some real-life circumstances of privately provided public goods for which the strict-normality assumption is not justified, at least not for all wealth levels.

One may even go one step further to say that certain types of public goods seem to assume the characteristics of an inferior good. For instance, public parks and public transportation (such as buses) in many metropolitan areas are majorly utilized by low-income people, or even the homeless. Many community facilities, which are open to the public and free to use, often have a private, paid-version substitute that dominates in terms of equipment and services. Examples include the community recreation center and a private gym, public tennis courts and private tennis clubs, residential and commercial playgrounds for children, etc. There is often a tendency for people to substitute private services for public facilities as they become wealthier.

In this section, we assume the public good is inferior for all consumers at all wealth levels. Even with the preceding examples, this global inferiority may seem too strong on the practical ground, but it largely simplifies the analysis and provides some interesting results.

(A3') *(Inferiority of the public good)* $U_1U_{21} - U_2U_{11} < 0$.[5]

Lemma 3. *If A2 and A3' hold, then (i) the public good is inferior, and (ii) the utility function* $U(\frac{1}{p}(w + y_{-i} - z), f(z))$ *has the strict SCP in* $(z; -y_{-i})$.

The inferiority of the public good requires (A3') $U_1U_{21} - U_2U_{11} < 0$, or the marginal rate of substitution, $U_2(x_i, q)/U_1(x_i, q)$, to be decreasing in x_i. Note that A3' is more likely to hold

[5] In addition, it is assumed that the consumer is wealthy enough such that at the lowest wealth level, her *standalone contribution* (as called by Varian, 1994), i.e., the contribution made when other agents contribute zero, is strictly between $(0, w)$, so that the consumer consumes positive amounts of both private and public goods. This avoids the possibility of an increasing part of the demand for the public good at low wealth levels, where the player spends all wealth on the inferior public good, which violates global inferiority.

when the utility function is convex in the private good, i.e., $U_{11} > 0$. Indeed, if the utility function is separable, i.e., $U_{21} = 0$, then A3' coincides with the convexity of the utility function in the private good.

By Lemma 3, when the public good is inferior, the utility function $U(\frac{1}{p}(w+y_{-i}-z), f(z))$ has the strict SCP in $(z; -y_{-i})$, which means that the player's reaction correspondence $z^*(y_{-i}; w)$ in terms of maximization (2) has a tendency to decrease in y_{-i}. However, because the feasibility constraint $z \in [y_{-i}, y_{-i} + w]$ is ascending in y_{-i}, the Monotonicity Selection Theorem cannot be directly applied. It is proved in the Appendix that every selection of the reaction correspondence $z^*(y_{-i}; w)$ is a (strictly) decreasing function of y_{-i}, until it hits the lower bound and slopes upward thereafter. In view of the relationship between $z^*(\cdot)$ and $y^*(\cdot)$, it suggests that every selection of $y^*(y_{-i}; w)$ is decreasing in y_{-i} quite rapidly with slopes no greater than -1, before it hits zero and stays thereafter (see Lemma 4 in Appendix for a full characterization of the graph of $y^*(y_{-i}; w)$ in this case).

As both $z^*(y_{-i}; w)$ and $y^*(y_{-i}; w)$ are decreasing in y_{-i}, it is well-known that there is no general result for the existence of Nash equilibrium in submodular games (with some exceptions, see e.g., Vives, 1999).[6] Indeed, symmetric equilibria may fail to exist in our model, but the existence is guaranteed by a "monopoly provision" equilibrium. The following Proposition characterizes all possible equilibria in this game.

Proposition 4. *With the normality of the private good (A2) and the inferiority of the public good (A3'), the following holds:*

1. There always exists an equilibrium where one consumer contributes a positive amount to supply the public good and other $(n-1)$ consumers do not contribute at all.

2. Whenever a symmetric equilibrium exists in an m-player game for some $m < n$, there is no other symmetric equilibrium, and it also constitutes an equilibrium for the n-player game, with the other $(n-m)$ consumers not contributing at all.

3. No other equilibrium exists except the ones characterized above, i.e., no such equilibrium exists where two consumers make positive yet different contributions to the public good.

[6] Novshek (1985) provides a fixed-point argument for Cournot competition when the best reply of each player is decreasing and only depends on the aggregate action of the rivals, which fits the context at hand and thus guarantees the existence of a Nash equilibrium (altenatively, see Thoerem 2.7 in Vives, 1999). A "monopoly provision" equilibrium always exists but other equilibria such as a symmetric one may not exist.

The "monopoly provision" equilibrium refers to one where one consumer contributes the "monopoly amount", i.e., the amount she contributes when she is the sole player of the game (the standalone contribution), while all other consumers free ride. There is a slight abuse of language here as the context has nothing to do with oligopoly competition, although it has been mentioned earlier that there is a certain connection between Cournot oligopoly and the public good game at hand. The contributor and free riders are mutually best-responding to each other because $y^*(y_{-i}; w)$ decreases at slopes no greater than -1, i.e., if y_{-i} increases by Δy, the best response of player i will decrease by more than Δy. This implies that the best response to a "monopoly provision" contribution y_i^1 is zero given that y_i^1 is best responding to zero. Similarly, the best response by a free rider to a m-player total contribution my_i^m is also zero, given that y_i^m is a player's best response to $(m-1)y_i^m$, which corresponds to the second half of part 2. In this case, the equilibrium is invariant to additional players who join and free ride (thus invariant to n), as long as the m contributors constitute a symmetric equilibrium among themselves, the latter may be termed as an m-player symmetric equilibrium.

Note that the m-player symmetric equilibrium may fail to exist as the reaction curve may have downward jumps and fail to possess a fixed point. Nevertheless, upon existence, it is unique in the sense that there is no other symmetric equilibrium in the m-player game, otherwise one can find a contradiction towards the fact that $y^*(y_{-i}; w)$ is decreasing in y_{-i}. (But there possibly exists other partially symmetric equilibrium with m' contributors and $m - m'$ free riders for some $m' < m$.) Moreover, all possible equilibria must take the partially symmetric form, i.e., all contributors must donate the same amount but free riders are allowed. If two contributors donate different amounts to the public good, it contradicts the fact that $z^*(y_{-i}; w)$ is strictly decreasing in y_{-i} (before it hits the lower bound) implied by A3'.

In a nutshell, the way to find all equilibria of a n-player game (apart from the monopoly-provision one) is to find all intersections of the decreasing reaction correspondence $y^*(y_{-i}; w)$ and the line $y_{-i}/(m-1)$ for each integer $m \in [2, n]$, each intersection defining a symmetric equilibrium for the m-player game and thus a partially symmetric equilibrium for the n-player game with $n - m$ free riders. The two graphs may never interest for any $m \in [2, n]$ due to incontinuity of $y^*(y_{-i}; w)$, but no other equilibrium exists apart from ones found in this way.

Proposition 5. *Under the same assumptions of Proposition 4, suppose a symmetric equilibrium* *exists in both an m-player and an n-player game, $n > m$, then*

$$y_{-i}^n > y_{-i}^m, \quad \textit{and} \;\; y_i^n < y_i^m, \;\; z^n < z^m, \;\; V^n > V^m.$$

The major change in the comparative statics compared to the normal public good case, is that $y^*(y_{-i}; w)$ decreases in y_{-i} so rapidly that even the equilibrium total contribution z^n decreases in n! Since the m-player symmetric equilibrium also constitutes a partially symmetric equilibrium for the n-player game, a direct consequence is that among all possible equilibria in an n-player game, surprisingly, the "monopoly provision" equilibrium has the highest public good supply. The free-riding problem becomes so severe when the public good is inferior that an extreme outcome as such emerges in the equilibrium. The next Corollary shows that if the utility function is convex in the private good, the "monopoly provision" equilibrium also delivers the highest social welfare.

Definition 3. *Define the social welfare as the sum of each consumer's utility in the game,* $W = \sum_{i=1}^n U(\frac{1}{p}(w - y_i), f(z))$.

Corollary 6. *Under the same assumptions of Proposition 4 and an additional assumption that* $U_{11} > 0$*, the "monopoly provision" equilibrium has the highest social welfare among all possible equilibria.*

If the utility function is separable (such as the one given in the following Example), i.e., $U_{21} = 0$, then $U_{11} > 0$ coincides with assumption A3' for the inferiority of the public good (the same observation is made by Liebhafsky, 1969). Note that the "monopoly contributor" is worse off than those contributors in an m-contributor partially symmetric equilibrium because $V^m > V^1$, but the free riders in the "monopoly provision" equilibrium are better off than those free riding in the m-contributor equilibrium because $z^m < z^1$. Corollary 6 settles that the gain of the latter outweighs the loss of the former given the convexity of the utility function in the private good. Although the "monopoly" public good provider would prefer an equilibrium with more contributors than herself alone, the latter may not exist, and even upon existence the cooperation may fail due to the free-riding incentives possessed by the other players.

Even with an inferior public good and an atypical Pareto-dominant "monopoly provision" equilibrium, the usual under-production problem associated with the private provision of public good may persist. In other words, the "monopoly provision" equilibrium may not be Pareto efficient. To see this, suppose for simplicity that the utility function is separable, $f(z) = z$, and the price of the private good equals 1. The conditions for the normality of the private good and inferiority of the public good, A2 and A3', become $U_{22} < 0$ and $U_{11} > 0$. The first-order condition of (1) for a free rider in a "monopoly provision" equilibrium suggests $-U_1(w, z^1) + U_2(w, z^1) < 0$, where z^1 is the amount donated by the monopoly contributor. Now suppose each of the $(n-1)$ free riders voluntarily donate dy while the monopoly contributor still donates z^1. The monopoly contributor is strictly better off because the new total public good supply has increased to $z^1 + (n-1)dy$. By committing to simultaneously increasing their contribution, the $(n-1)$ free riders are also better off if $-U_1(w, z^1) + (n-1)U_2(w, z^1) > 0$, which is compatible with other conditions. In fact, the Pareto improvement happens almost inevitably as the group size n increases.

There are very few examples of an inferior public good that is privately provided by a single agent. Still, one may think of some community goods that fit with the story. For instance, the portable library (contained in a small wooden box) seen at the edge of a private garden for public use, is likely built by the house owner alone with very little collaboration from the neighborhood. In most cases, these community facilities are built under the government's intervention. There may be a reason why the private provision of inferior public goods is so rare to see. If "monopoly provision" is the unique equilibrium outcome, there is then a role assignment issue as to which agent should supply the public good, and the agents may be well reluctant to do so given the extremeness of the provision prospect (i.e., knowing all other agents will free ride on her) out of fairness concern, possibly resulting in no provision at all. This is another justification for government's intervention in such a case except for the usual reason to overcome the under-production problem.

Example 2. Suppose the utility function has the form $U(x_i, q) = \ln(q - a) - 2\ln(b - x_i)$ where $q = f(z) = z$. Given y_{-i}, consumer i maximizes her utility subject to constraints:

$$\max_{x_i, y_i} \; U = \ln(y_i + y_{-i} - a) - 2\ln(b - x_i)$$

$$s.t. \;\; px_i + y_i = w, \; 0 < x_i < b, \; y_i > \max\{0, a\}$$

Since the utility function is separable and the production function is the identity one, A2 coincides with U being concave in q and A3' with U being convex in x_i, both satisfied. Solving the standalone consumer problem be letting $y_{-i} = 0$, the demand for the public good is $y^* = 2a + bp - w$ and the demand for the private good is $x^* = (2w - 2a - bp)/p$, thus the public good is inferior and private good is normal.

Besides the budget constraint $y_i \in [0, w]$, the logarithm requires $y_{-i} > a$ (in case $y_{-i} = 0$) and $y_{-i} > w - bp$ so that $x_i < b$. Substitute y_i for x_i by the budget constraint and solve the utility maximization, giving rise to the first-order condition

$$y^*(y_{-i}; w) = 2a + bp - w - 2y_{-i}, \tag{4}$$

which is decreasing in y_{-i}. The symmetric equilibrium is where $y^*(y_{-i}; w) = y_{-i}/(n-1)$. It can be verified that when $a \in (-\frac{bp}{2}, 0)$ and $w \in (a + \frac{bp}{2}, 2a + bp)$, the game has an interior symmetric equilibrium where $y^n_{-i} = \frac{n-1}{2n-1}(2a+bp-w)$, $y^n_i = \frac{1}{2n-1}(2a+bp-w)$, and $z^n = \frac{n}{2n-1}(2a+bp-w)$. Comparative statics is as expected: y^n_{-i} is increasing in n, both y^n_i and z^n decreasing in n.

Because the reaction curve (4) is continuous, the n-player game also has all the partially symmetric equilibrium indexed by the number of contributors m, where each contributor in an m-contributor equilibrium contributes $y^m_i = \frac{1}{2m-1}(2a + bp - w)$, while others free ride. In particular, in the "monopoly provision" equilibrium the sole contributor contributes $y^1_i = 2a + bp - w$. It follows immediately from (4) that $y^*(y^1_i; w) = -y^1_i < 0$, thus other players best respond by not contributing at all. And one can verify that the "monopoly provision" equilibrium also delivers the highest welfare among all equilibria. Specifically, in this example the equilibrium welfare is reversely ranked by the number of contributors in the equilibrium, i.e., the "monopoly provision" equilibrium generates higher welfare than the two-contributor equilibrium, which generates higher welfare than the three-contributor equilibrium. The last result cannot be generalized for an arbitrary seperable utility function.

5 Conclusion

This paper studies the group size effects in a simple model of the private provision of public goods. The use of new tools from the lattice theory allows to discard unnecessary conditions used in traditional methods for the existence and uniqueness of Nash equilibrium, in particular the strict quasi-concavity of the utility function. Using a generalized version of Topkis' (1978) results for supermodular games, namely the single-crossing property of a function and the Monotone Selection Theorem proposed in Milgrom and Shannon (1994), the model gives rise to general, unambiguous comparative statics conclusions with a minimal set of conditions, which are related to the normality/inferiority of the public and private goods. It is shown that the incentive to free ride increases with the group size as long as the private good is normal, in the sense that the (extremal) equilibrium individual contribution to the public good decreases with n. Though the total contribution may still increase with n when the public good is normal. If the public good is inferior, an extreme situation arises where even the (extremal) equilibrium total contribution decreases in n. It implies that a "monopoly provision" equilibrium may lead to the highest amount of public good supply among all possible equilibria, and it also generates the highest social welfare if the utility function is convex in the private good.

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Appendix. Proofs

This section provides all the proofs of the paper. Some important notations are introduced first. A consumer i's best-response correspondence is defined by

$$y^*(y_{-i};w) = \arg\max\{U(\frac{1}{p}(w-y_i), f(y_i+y_{-i})) : 0 \le y_i \le w\}, \tag{5}$$

where $y_{-i} \in [0,(n-1)w]$.

Alternatively, one may think of a consumer as choosing the total contribution z of the public good, given $(n-1)$ other consumers' total contribution y_{-i}. So the best-response correspondence in (5) can be rewritten in terms of z as follows:

$$z^*(y_{-i};w) = \arg\max\{U(\frac{1}{p}(w+y_{-i}-z), f(z)) : y_{-i} \le z \le y_{-i}+w\}. \tag{6}$$

As all consumers are identical, each consumer has the same utility function, so that their best-response correspondences $y^*(y_{-i};w)$ and $z^*(y_{-i};w)$ are also the same. Then, as in Amir and Lambson (2000), one has the following mapping based on the best-response correspondence (5):

$$B_n : [0,(n-1)w] \to 2^{[0,(n-1)w]},$$
$$y_{-i} \to \frac{n-1}{n}(y_i' + y_{-i}),$$

where y_i' denotes a best-response level of consumer i's contribution of the public good when the total contribution of other $(n-1)$ consumers is y_{-i}, i.e., $y_i' \in y^*(y_{-i};w)$. The combined constraints $y_i \in [0,w]$ and $y_{-i} \in [0,(n-1)w]$ guarantee that B_n maps some y_{-i} into the same space $[0,(n-1)w]$. The mapping B_n is of particular importance while dealing with symmetric equilibria. Indeed, any fixed-point of B_n yields a symmetric Nash equilibrium, as it satisfies $y_{-i} = \frac{n-1}{n}(y_i' + y_{-i})$, or $y_i' = \frac{1}{n-1}y_{-i}$, which means that every consumer contributes the same amount of the public good.

Proof of Lemma 1. Recall that the utility function can be written in terms of x_i as $U(x_i, f(w - px_i + y_{-i}))$ where $w - px_i = y_i$. Following Milgrom and Shannon (1994), I use the method of dissection to prove the two parts stated in the Lemma. Define $\tilde{U}(x, y, t) = U(x, f(t+y))$. $\tilde{U}$ is completely regular, i.e., $\tilde{U}_y = U_2 f' > 0$. Simple calculus gives $\tilde{U}_x/|\tilde{U}_y| = \frac{U_1}{U_2 f'}$, which is strictly increasing in t if and only if $(f')^2(U_2U_{21} - U_1U_{22} - \frac{f''}{(f')^2}U_1U_2) > 0$, which is true by Assumption A2. Therefore, $\tilde{U}$ satisfies the strict Spence-Mirrlees condition.

To prove part (i), let $y = h(x) = -px + y_{-i}$ where $h(x) = -px + y_{-i}$ belongs to the richly parameterized family $\{\alpha_1 x + \alpha_0 : \alpha_1, \alpha_0 \in \mathbb{R}\}$ with $\alpha_1 = -p$ and $\alpha_0 = y_{-i}$. By [Milgrom and Shannon (1994), Theorem 11], since $\tilde{U}$ satisfies the strict Spence-Mirrlees condition, $U(x, f(t - px + y_{-i}))$ satisfies the strict SCP in $(x; t)$. Now with the change of variable by letting $x = x_i$ and $t = w$, it follows that $U(x_i, f(w - px_i + y_{-i}))$ has the strict SCP in $(x_i; w)$. Since the constraint set $x \in [0, \frac{w}{p}]$ is ascending in w, by the Monotone Selection Theorem (Milgrom and Shannon, 1994), for every y_{-i}, every selection of $x^*(y_{-i}; w) = \arg\max\{U(x_i, f(w - px_i + y_{-i})) : 0 \leq x_i \leq \frac{w}{p}\}$ is increasing in w. Therefore, it is verified that the private good is normal.

To prove part (ii), let $y = h(x) = -px + w$ which belongs to the richly parameterized family $\{\alpha_1 x + \alpha_0 : \alpha_1, \alpha_0 \in \mathbb{R}\}$ with $\alpha_1 = -p$ and $\alpha_0 = w$. By [Milgrom and Shannon (1994), Theorem 11], since $\tilde{U}$ satisfies the strict Spence-Mirrlees condition, $U(x, f(t - px + w))$ satisfies the strict SCP in $(x; t)$. Now let $x = \frac{1}{p}(w - y_i)$ and $t = y_{-i}$, it follows that $U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ has the strict SCP in $(\frac{1}{p}(w - y_i); y_{-i})$. Since $\frac{1}{p}(w - y_i)$ is a monotone (linear) transformation of $-y_i$, it follows by the definition of SCP that for every p and w, $U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ has the strict SCP in $(-y_i; y_{-i})$. Because the constraint set $y_i \in [0, w]$ is constant with respect to y_{-i}, by the Monotonicity Selection Theorem (Milgrom and Shannon, 1994), every selection of $-y^*(y_{-i}; w)$ is increasing in y_{-i}. Then (ii) follows. Q.E.D.

Remark. Note that if $U_2U_{21} - U_1U_{22} - \frac{f''}{(f')^2}U_1U_2 < 0$ (reversed inequality of A2) and A3 (normality of the public good) hold, then the conclusions of Lemma 1 reverse, i.e., the private good is inferior and the utility has the strict SCP in $(y_i; y_{-i})$. The idea is that when $U_2U_{21} - U_1U_{22} - \frac{f''}{(f')^2}U_1U_2 < 0$, by checking $\tilde{U}_x/|\tilde{U}_y|$, one concludes that $\tilde{U}'(x, y, t) = U(x, f(-t + y))$ satisfies the strict Spence-Mirrlees condition (with a reversed order on t).

Then by letting $y = h(x) = -px + y_{-i}$, $x = x_i$ and $t = -w$, if follows that the utility function $U(x_i, f(-(-w) - px_i + y_{-i}))$ satisfies the strict SCP in $(x_i; -w)$. This means that $x^*(y_{-i}; w)$ has a tendency to be decreasing in w (increasing in $-w$), but the Monotonicity Selection Theorem cannot be directly applied, because the feasibility constraint of x_i, $[0, w/p]$ is ascending in w. Nevertheless, it can still be concluded that every selection of $x^*(y_{-i}; w)$ is decreasing in w along any interior and continuous part of its graph.[7] For the discontinuous part, every selection of $x^*(y_{-i}; w)$ can only jump downward as w increases. This is true under A3, because then the public good is normal by Lemma 2, hence $y^*(y_{-i}; w) = w - px^*(y_{-i}; w)$ can only jump upward as w increases at any discontinuous points.

The arguments provided in the preceding paragraph jointly imply that as long as the graph of $x^*(y_{-i}; w)$ starts at an interior point when $w = \underline{w}$, i.e., $x^*(y_{-i}; \underline{w}) < \underline{w}$, then its graph will either decrease continuously or jump downward as w increases until it hits zero. Indeed, $x^*(y_{-i}; \underline{w}) < \underline{w}$ is true by A3'.[8] And one concludes the normality the the private good.

Similar to Lemma 1, the two conditions also imply that every selection of the reaction correspondence $y^*(y_{-i}; w)$ is increasing in y_{-i}. The formal proof of the latter is omitted here, but it is rather procedural, because the constraint $y_i \in [0, w]$ is constant with respect to y_{-i}, so that the Monotonicity Selection Therorem can be applied directly.

Proof of Lemma 2. I want to show that the condition $U_1U_{21} - U_2U_{11} > 0$ (A3) implies (i) $U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ has the SCP in $(y_i; w)$, which further implies the normality of y_i, and (ii) $U(\frac{1}{p}(w + y_{-i} - z), f(z))$ has the SCP in $(z; y_{-i})$, which further implies $z^*(y_{-i}; w)$ is increasing in y_{-i}. Following Milgrom and Roberts (1994), I prove this by the method of dissection.

Define $\tilde{U}(x, y, t) = U(\frac{1}{p}(t - x), y)$ as a parameterized function of (x, y) (the notation (x, y, t) used here follows Milgrom and Roberts (1994), which has nothing to do with the notation for the private good and public good). $\tilde{U}$ is completely regular, i.e., $\tilde{U}_y = U_2 > 0$. Simple calculus gives $\tilde{U}_x/|\tilde{U}_y| = \frac{-U_1}{pU_2}$, which is strictly increasing in t if and only if $U_1U_{21} - U_2U_{11} > 0$ (A3). Therefore, $\tilde{U}$ satisfies the strict Spence-Mirrlees condition.

[7] This argument uses the notion of Rectangle Monotonicity given in [Amir and Lambson (2000), Proof of Lemma 3.1]. The formal proof is omitted here, but is analogous to that included in the Proof of Lemma 3, step (a), to justify the inferiority of the public good, where the Rectangle Monotonicity for $y^*(y_{-i}; w)$ is formally stated and proved.

[8] In the footnote attached to A3', it is assumed that $x^*(0; \underline{w}) < \underline{w}/p$, which implies that $x^*(y_{-i}; \underline{w}) < \underline{w}/p$. To see this, first inspect the utility function $U(x_i, f(w - px_i + y_{-i}))$, and notice that the parameters w and y_{-i} are mathematically equivalent when maximizing over x_i, hence the utility function also satisfies the strict SCP in $(x_i; -y_{-i})$ using an analogous proof of that for $(x_i; -w)$. Since the constraint of x_i, $[0, w/p]$, is constant with respect to y_{-i}, there is no trouble to apply the Monotonicity Selection Theorem and conclude that every selection of $x^*(y_{-i}; w)$ is decreasing in y_{-i}. It follows that $x^*(y_{-i}; \underline{w}) \leq x^*(0; \underline{w}) < \underline{w}/p$ for any $y_{-i} > 0$.

To prove part (i), let $y = h(x) = f(x + y_{-i})$ where $h(x) = f(x + y_{-i})$ belongs to the richly parameterized family $\{f(x + \alpha_2) + \alpha_1 x + \alpha_0 : \alpha_2, \alpha_1, \alpha_0 \in \mathbb{R}\}$ with $\alpha_1 = \alpha_0 = 0, \alpha_2 = y_{-i}$. By [Milgrom and Shannon (1994), Theorem 11], $\tilde{U}(x, h(x), t) = U(\frac{1}{p}(t - x), f(x + y_{-i}))$ satisfies the strict SCP in $(x; t)$. By change of variable, let $x = y_i$ and $t = w$, and it follows that $U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ has the strict SCP in $(y_i; w)$. Now that constraint set $y_i \in [0, w]$ is ascending in w, and $U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ has the strict SCP in $(y_i; w)$, by the Monotone Selection Theorem (Milgrom and Shannon, 1994), for every y_{-i}, every selection of $y^*(y_{-i}; w)$ given in (5) is increasing in w, so the public good is normal.

To verify part (ii), first let $y = h(x) = f(x)$ where $h(x) = f(x)$ belongs to the richly parameterized family $\{f(x) + \alpha_1 x + \alpha_0 : \alpha_1, \alpha_0 \in \mathbb{R}\}$. Because $\tilde{U}$ satisfies the strict Spence-Mirrlees condition, by [Milgrom and Shannon (1994), Theorem 11], $\tilde{U}(x, h(x), t) = U(\frac{1}{p}(t - x), f(x))$ satisfies the strict SCP in $(x; t)$. Now let $x = z$ and $t = w + y_{-i}$, it follows that $U(\frac{1}{p}(w + y_{-i} - z), f(z))$ has the strict SCP in $(z; w + y_{-i})$. Fixing w, it follows (by the definition of SCP) that $U(\frac{1}{p}(w + y_{-i} - z), f(z))$ has the strict SCP in $(z; y_{-i})$. Notice that the constraint set $z \in [y_{-i}, y_{-i} + w]$ is also strictly ascending in y_{-i}. By the Monotone Selection Theorem, every selection of $z^*(y_{-i}; w)$ is increasing in y_{-i}, for any given w. Q.E.D.

Proof of Proposition 1

Consider the utility maximization (6). First, I show that a symmetric equilibrium exists. By Lemma 2, every selection of $z^*(y_{-i})$ is increasing in y_{-i} (the argument w is omitted for simplicity whenever there is no risk of confusion). Since for any selection $y' \in y^*(y_{-i})$, there exists some selection $z' \in z^*(y_{-i})$ such that $z' = y' + y_{-i}$, it follows that every selection of B_n is increasing in y_{-i} for any fixed n. By Tarski's fixed-point theorem, B_n has a fixed point, which is a symmetric Nash equilibrium by construction.

To prove that no asymmetric equilibrium exists, it is sufficient to show that every selection of $z^*(y_{-i})$ is strictly increasing in y_{-i}. Indeed, this would mean that at most one y_{-i} corresponds to each $z' \in z^*(y_{-i})$, s.t. $z' = y'_i + y_{-i}$, with y'_i being the best-response to y_{-i}. But then, for each total contribution z' of public good, each consumer would contribute the same level of public good $y'_i = z' - y_{-i}$, where $y_{-i} = (n - 1)y'_i$, implying symmetry in the equilibrium.

Consider an arbitrary selection of $z^*(y_{-i})$, denoted by $\tilde{z}$. To prove that the mapping $y_{-i} \to z^*(y_{-i})$ is strictly increasing, let us assume the contrary: There exist some y_1 and y_2, with

$y_1 > y_2$, such that $\tilde{z}(y_1) = \tilde{z}(y_2)$, since every selection of $z^*(y_{-i})$ has been proved to be (weakly) increasing in y_{-i}. Indeed, $\tilde{z}(y_{-i})$ will be constant for all $y_{-i} \in [y_2, y_1]$, thus both $\tilde{z}(y_1)$ and $\tilde{z}(y_2)$ can be without loss of generality taken to be interior solutions to (6), so each of them satisfies the F.O.C.,

$$U_1^i(\frac{1}{p}(w - z + y_j), f(z))(-\frac{1}{p}) + U_2^i(\frac{1}{p}(w - z + y_j), f(z))f'(z) = 0, \quad j = 1, 2,$$

where $z \equiv \tilde{z}(y_1) = \tilde{z}(y_2)$. Then the F.O.C. implies:

$$\begin{aligned} U_1^i(\frac{1}{p}(w - z + y_1), f(z))(-\frac{1}{p}) + U_2^i(\frac{1}{p}(w - z + y_1), f(z))f'(z) \\ = U_1^i(\frac{1}{p}(w - z + y_2), f(z))(-\frac{1}{p}) + U_2^i(\frac{1}{p}(w - z + y_2), f(z))f'(z), \end{aligned}$$

or

$$\begin{aligned} -\frac{1}{p}\frac{U_1^i(\frac{1}{p}(w - z + y_1), f(z)) - U_1^i(\frac{1}{p}(w - z + y_2), f(z))}{\frac{1}{p}(y_1 - y_2)} \\ + f'(z)\frac{U_2^i(\frac{1}{p}(w - z + y_1), f(z)) - U_2^i(\frac{1}{p}(w - z + y_2), f(z))}{\frac{1}{p}(y_1 - y_2)} = 0. \end{aligned}$$

This holds for all $y_{-i} \in [y_2, y_1]$. Hence, I can take a limit as $y_2 \to y_1$ (so $\frac{1}{p}y_2 \to \frac{1}{p}y_1$), and it follows

$$-\frac{1}{p}U_{11}^i + U_{21}^i f' = 0 \text{ at } (y_1, z). \tag{7}$$

But this is easily seen to violate Assumption A3, because if one replaces p by the F.O.C., $-\frac{1}{p}U_1 + U_2 f' = 0$, (7) implies $U_2 U_{11} - U_1 U_{21} = 0$, contradicting A3. This leads us to the conclusion that $\tilde{z}(y_{-i})$ is strictly increasing and thus no asymmetric equilibrium exists. Q.E.D.

Proof of Proposition 2

1. Consider the mapping introduced above:

$$\begin{aligned} B_n : [0, (n-1)w] \to 2^{[0,(n-1)w]}, \\ y_{-i} \to \frac{n-1}{n}(y_i' + y_{-i}). \end{aligned}$$

By the Monotonicity Theorem (Milgrom and Shannon, 1994), the fact that the utility function is continuous and has the SCP in $(z; y_{-i})$, and the action set $[y_{-i}, y_{-i} + w]$ is compact and ascending implies that the maximal and minimal selections of the arg max $z^*(y_{-i})$ given in (6), $\bar{z}(y_{-i})$ and $\underline{z}(y_{-i})$ exist (the maximizer is nonempty by continuity and compactness). It means that the maximal and minimal selections of B_n, denoted by $\bar{B}_n$ and $\underline{B}_n$ respectively, also exist. And it follows from the construction of B_n that the largest equilibrium value of the joint contribution of $n-1$ players, $\bar{y}^n_{-i}$, is also the largest fixed point of $\bar{B}_n$. Since $\frac{n-1}{n}$ is increasing in n, $\bar{B}_n(y_{-i})$ is increasing in n for every fixed y_{-i}. Then the largest fixed point of $\bar{B}_n$, which is $\bar{y}^n_{-i}$, is increasing in n due to Milgrom and Roberts (1990) (treating n as a parameter). A similar argument establishes that $\underline{y}^n_{-i}$ is also increasing in n.

2. Since $\bar{y}^n_{-i}$ is increasing in n and every selection of $z^*(y_{-i})$ is increasing in y_{-i}, it follows that the largest total equilibrium contribution to the public good, $\bar{z}^n = \bar{z}(\bar{y}^n_{-i})$, is also increasing in n. The same arguments hold for the smallest total equilibrium contribution $\underline{z}^n = \underline{z}(\underline{y}^n_{-i})$.

3. By the Envelop Theorem, the indirect utility function, $V(w, y_{-i}) = \max_{0\leq y_i\leq w} U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$, is strictly increasing in other players' joint contribution y_{-i}, as

$$\frac{\partial V(w, y_{-i})}{\partial y_{-i}} = \frac{\partial U(\frac{1}{p}(w - y_i'), f(y_i' + y_{-i}))}{\partial y_{-i}} = U_2 f' > 0.$$

where $y_i' \in y^*(y_{-i}; w)$. Since $\bar{y}^n_{-i}$ and $\underline{y}^n_{-i}$ are increasing in n, the extremal values of the indirect utility function, $\bar{V}^n = V(w, \bar{y}^n_{-i})$ and $\underline{V}^n = V(w, \underline{y}^n_{-i})$ are also increasing in n. Q.E.D.

Proof of Proposition 3

By Lemma 1 and 2, the slope of every selection of $y^*(y_{-i})$ is between the interval $[-1, 0]$. Then the uniqueness of equilibrium follows from a quite standard argument given in Amir (1996a). Here an alternative proof is presented. By Proposition 1, no asymmetric equilibrium can exist. Assume towards contradiction that there exist two symmetric Nash equilibria, denoted by $(y, ..., y)$ and $(\hat{y}, ..., \hat{y})$, $y \neq \hat{y}$. Suppose $y > \hat{y}$, but then $(n-1)y > (n-1)\hat{y}$ where $y \in y^*((n-1)y)$ and $\hat{y} \in y^*((n-1)\hat{y})$, implying that the reaction curve is strictly increasing between the two points. This contradicts the fact that the slope of $y^*(y_{-i})$ is bounded above by 0, thus the equilibrium is unique.

Because every selection of $y^*(y_{-i})$ is decreasing in y_{-i}, and the equilibrium value of y^n_{-i} (the upper and lower bars dropped due to uniqueness) is increasing in n by Proposition 2, then the individual contribution to the public good, $y_i^n \in y^*(y_{-i}^n)$ is decreasing in n. The opposite holds for the private good consumption as $x_i^n = \frac{1}{p}(w - y_i^n)$. Q.E.D.

Proof of Lemma 3. The proof uses the method of dissection and is analogous to that used for Lemma 2. Define $\tilde{U}(x, y, t) = U(\frac{1}{p}(-t - x), y)$. Simple calculus gives $\tilde{U}_x/|\tilde{U}_y| = \frac{-U_1}{pU_2}$, which is strictly increasing in t if and only if $U_1U_{21} - U_2U_{11} < 0$ (A3'). Therefore, $\tilde{U}$ satisfies the strict Spence-Mirrlees condition.

I first prove part (ii). Let $y = h(x) = f(x)$. By [Milgrom and Shannon (1994), Theorem 11], $\tilde{U}(x, h(x), t) = U(\frac{1}{p}(-t - x), f(x))$ satisfies the strict SCP in $(x; t)$. Now let $x = z$ and $t = -(w + y_{-i})$, it follows that $U(\frac{1}{p}(w + y_{-i} - z), f(z))$ has the strict SCP in $(z; -(w + y_{-i}))$. Fixing w, it follows (by the definition of SCP) that $U(\frac{1}{p}(w + y_{-i} - z), f(z))$ has the strict SCP in $(z; -y_{-i})$.

Going back to part (i), let $y = h(x) = f(x + y_{-i})$. By [Milgrom and Shannon (1994), Theorem 11], $\tilde{U}(x, h(x), t) = U(\frac{1}{p}(-t - x), f(x + y_{-i}))$ satisfies the strict SCP in $(x; t)$. Now let $x = y_i$ and $t = -w$, and it follows that $U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ has the strict SCP in $(y_i; -w)$. I want to show that the arg max of (5), $y^*(y_{-i}; w)$ is decreasing in w (in terms of each of its selection), but the Monotone Selection Theorem (Milgrom and Shannon, 1994) cannot be directly applied here because the constraint set $y_i \in [0, w]$ is ascending in w. Instead I show it in two steps (just as the way I dealt with the private good in the Remark of Lemma 1): (a) every selection of $y^*(y_{-i}; w)$ must be decreasing in w along any interior continuous part of itself and, (b) it cannot have upward jumps.

Step (a) uses the idea of *Rectangle Monotonicity* discussed in [Amir and Lambson (2000), Proof of Lemma 3.1]. Formally, suppose (a) does not hold, then there exists a selection of $y^*(y_{-i}; w)$, part of which is interior with respect to the constraint set and is continuously increasing in w. Consequently, there must exist two points (supposedly close to each other) on this part of $y^*(y_{-i}; w)$ such that the rectangle enclosing the graph between the two points is fully inscribed in the constraint set, i.e., $y_1 \in y^*(y_{-i}; w_1)$ and $y_2 \in y^*(y_{-i}; w_2)$ such that $w_1 < w_2$, $y_1 < y_2$, and $y_1 \in [0, w_2]$, $y_2 \in [0, w_1]$. Note that both continuity and interiority are crucial to finding two such points. But then there is a contradiction. To see this, drop the argument y_{-i} and write $U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ as $\hat{U}(y_i; w)$ for simplicity. Then $\hat{U}(y_2; w_2) \geq \hat{U}(y_1; w_2)$

because $y_2 \in y^*(y_{-i}; w_2)$, but then because $U(\frac{1}{p}(w - y_i), f(y_i + y_{-i}))$ has the strict SCP in $(y_i; -w)$ and $-w_2 < -w_1$, it follows that $\hat{U}(y_2; w_1) > \hat{U}(y_1; w_1)$, which contradicts the fact that $y_1 \in y^*(y_{-i}; w_1)$. The proof of step (a) is thus complete. Step (b) is a direct consequence of Assumption A2, which leads to the normality of the private good by Lemma 1. Therefore, $x^*(y_{-i}; w)$ can only have upward jumps in w, and thus $y^*(y_{-i}; w) = w - px^*(y_{-i}; w)$ can only have downward jumps.

Now by assumption A3' (see footnote), the graph of $y^*(y_{-i}; w)$ starts at some interior point(s) at the lowest wealth level, and is continuously decreasing possibly with downward jumps, until it hits zero where it stays thereafter. Therefore, the public good is inferior and the proof of part (i) is complete. Q.E.D.

The following Lemma is crucial to proving Proposition 4, which gives a full characterization of the graph of a player's reaction correspondence $y^*(y_{-i})$. With the normality of the private good and the inferiority of the public good, $y^*(y_{-i})$ must be decreasing at slopes no more than -1 and can only have downward jumps at the discontinuous points. The arguments are similar to that used in the proof of Lemma 3, where $y^*(y_{-i}; w)$ is shown to be decreasing in w despite of the ascending constraint set $[0, w]$. The idea is to use the notion of *Rectangle Monotonocity* given in Amir and Lambson (2000), thus any continuous part of $y^*(y_{-i})$ must be decreasing at slopes less than -1, and the fact that every selection of $y^*(y_{-i})$ must be decreasing by A2.

Lemma 4. *If A2 and A3' hold, then every selection of the reaction correspondence, $\tilde{y} \in y^*(y_{-i})$, (i) must have slopes no greater than -1 along any continuous, interior (below w) part of its graph, (ii) cannot have upward jumps.*

Proof. Pick an arbitrary selection $\tilde{y}(y_{-i}) \in y^*(y_{-i})$. By Lemma 1, every selection of $y^*(y_{-i})$ is decreasing, hence $\tilde{y}(y_{-i})$ cannot have upward jumps, which proves part (ii).

To show part (i), one can equivalently show that $\tilde{z}(y_{-i}) = \tilde{y}(y_{-i}) + y_{-i}$ is decreasing in y_{-i} along any continuous part of its graph that is interior with respect to the constraint set $z \in [y_{-i}, w + y_{-i}]$. By Lemma 3, A3' implies that $U(\frac{1}{p}(w + y_{-i} - z), f(z))$ has the strict SCP on $(z; -y_{-i})$, indicating a tendency of z^* to decrease in y_{-i}. Because the constraint set $z \in [y_{-i}, y_{-i} + w]$ is ascending in y_{-i}, the Monotone Selection Theorem (Milgrom and Shannon, 1994) cannot be directly applied. However, one can show instead that any interior, continuous part of the graph of $\tilde{z}(y_{-i})$ must be decreasing in y_{-i} (i.e., corresponding to the Rectangle Monotonicity property as in Amir and Lambson, 2000). Assume towards contradiction that an interior continuous part of $\tilde{z}(y_{-i})$ is not decreasing, then along this part of the curve there must exist two points $z_1 = \tilde{z}(y_1)$ and $z_2 = \tilde{z}(y_2)$, $y_1 < y_2$, supposedly close to each other such that the rectangular enclosing the graph of the two points is fully inscribed in the constraint set, i.e., $z_1 \in [y_2, y_2 + w]$ and $z_2 \in [y_1, y_1 + w]$. Note that both continuity and interiority are crucial to finding two such points. But then there is a contradiction. Drop the argument w and write $U(\frac{1}{p}(w + y_{-i} - z), f(z))$ as $\hat{U}(z; y_{-i})$. Then $\hat{U}(z_2; y_2) \geq \hat{U}(z_1; y_2)$ because $z_2 = \tilde{z}(y_2)$. But then because $\hat{U}(z; y_{-i})$ has the strict SCP in $(z; -y_{-i})$ and $-y_2 < -y_1$, it follows that $\hat{U}(z_2; y_1) > \hat{U}(z_1; y_1)$, contradicting the fact that $z_1 = \tilde{z}(y_1)$, which proves part (i).

Part (i) and (ii) allow us to fully characterize the graph of $\tilde{y}(y_{-i})$ at any wealth level w if it starts strictly below w when $y_{-i} = 0$, i.e., the standalone contribution is strictly below w, since neither (i) or (ii) rules out the possibility that $\tilde{y}(y_{-i}; w) = w$ for small y_{-i}'s (let us briefly recover the argument w in the reaction curve). For the standalone contribution to be strictly below w for any wealth level w, i.e., $\tilde{y}(0; w) < w$, one only needs it to hold at the lowest wealth level, say, $\underline{w}$, which is assumed true (see the footnote attached to A3'). Because then $\tilde{y}(0; w) \leq \tilde{y}(0; \underline{w}) < \underline{w} < w$, the first inequality derived from Lemma 3, i.e., the public good is inferior under A2 and A3'.

To summarize, any selection of $y^*(y_{-i}; w)$ starts at an interior point (below w) when $y_{-i} = 0$, and then either continuously decreases in y_{-i} with slopes no higher than -1 or has downward jumps at any discontinuous points, until it hits 0 and stays thereafter. Q.E.D.

Proof of Proposition 4

1. First, I want to prove the existence of a "monopoly provision" equilibrium where one consumer contributes to the public good and the other $(n-1)$ consumers do not contribute at all. Pick an arbitrary $y_0 \in y^*(0)$, which is the standalone contribution of a player, $y_0 > 0$ (validated by the footnote attached to A3'). For the "monopoly provision" scenario to be a Nash equilibrium, it suffices to show $y^*(y_0) = \{0\}$. Suppose not, that there is some $y_1 \in y^*(y_0)$ such that $y_1 > 0$, then there are two best-responding points $y_0 \in y^*(0)$ and $y_1 \in y^*(y_0)$, such that

$$\frac{y_1 - y_0}{y_0 - 0} = \frac{y_1}{y_0} - 1 > -1,$$

contradicting the fact that the slopes of $y^*(y_0)$ have slopes no greater than -1 by Lemma 4. Hence $(y_0, 0, ..., 0)$ is a Nash equilibrium.

2. Next, I want to show whenever a symmetric equilibrium exists in an m-player game, it must be unique. Indeed, the existence of a symmetric equilibrium is generally not guaranteed in submodular games (except in two-player games where submodular games can be turned into supermodular games by taking the reversed order of one player's action). On the graph, it is easily seen that $y^*(y_{-i})$ may fail to intersect the line $y_{-i}/(n-1)$ due to possible downward jumps, while the intersection defines a symmetric equilibrium. However, if such a symmetric equilibrium exists, it must be unique, because the strongly decreasing $y^*(y_{-i})$ (i.e., any of its selections has slopes no more than -1) and the upward-sloping line $y_{-i}/(n-1)$ can at most intersect once.

Now assume such a symmetric equilibrium exists in an m-player game, with each player contributing $y_m > 0$ to the public good, i.e., $y_m \in y^*((m-1)y_m)$. Then for the n-player game, $n > m$, I claim that m players each contributing y_m with the other $(n-m)$ players contributing nothing constitutes an equilibrium. It suffices to show $y^*(my_m) = \{0\}$. The proof is similar to part 1. Suppose not, so that there exists $y_2 > 0$ such that $y_2 \in y^*(my_m)$. Now consider the two points $y_m \in y^*((m-1)y_m)$ and $y_2 \in y^*(my_m)$, and it follows

$$\frac{y_2 - y_m}{my_m - (m-1)y_m} = \frac{y_2}{y_m} - 1 > -1,$$

contradicting the fact that $y^*(y_{-i})$ can only have slopes no greater than -1 asserted by Lemma 4. Thus it is proved that the m-player symmetric equilibrium also constitutes a partially symmetric equilibrium in an n-player game, with the other $(n-m)$ players contributing nothing.

3. Lastly, I want to show that there exists no other asymmetric equilibria except the ones mentioned in part 2. That is, in any Nash equilibrium, there are no two active contributors (i.e., with $y_i > 0$) who contribute different levels of the public good. In light of the characterization of the graph of $y^*(y_{-i})$ given in Lemma 4, it can be inferred that the graph of any selection of $z_i^*(y_{-i}) = y^*(y_{-i}) + y_{-i}$ must start at an interior point below w, decrease in y_{-i} with downward jumps allowed (call this the decreasing part), until it hits the lower bound y_{-i} and stays on it thereafter (call this the increasing part). The decreasing part is where the player makes

positive contributions (i.e., is active), and the increasing part is where the player contributes nothing. At this point, it suffices to show that every selection of $z^*(y_{-i})$ is strictly decreasing in y_{-i} along its decreasing part, because then there is only one value of y_{-i} corresponding to each value of z along the decreasing part, meaning the active players must contribute at the same level. The arguments needed to show strictly decreasing $z_i^*(y_{-i})$ are the same as those stated to show strictly increasing $z_i^*(y_{-i})$ in the Proof of Proposition 1: in the same manner, one needs to find two points to construct equation (7), which leads to a contradiction toward Assumptions A3'. The proof is omitted to conserve space. Q.E.D.

Proof of Proposition 5

A symmetric equilibrium is defined as the intersection of $y^*(y_{-i})$ and $y_{-i}/(n-1)$. Recall that by Lemma 4, $y^*(y_{-i})$ is strongly decreasing in the sense that every selection of it can only have slopes no greater than -1, and can only jump downward. When n increases, $y_{-i}/(n-1)$ shifts downwards (i.e., rotates clockwise), and if it still intersects with $y^*(y_{-i})$, the equilibrium value of y_{-i}^n must increase. That is, if the best-reply correspondence $y^*(y_{-i})$ intersects with both $y_{-i}/(m-1)$ and $y_{-i}/(n-1)$, $m < n$, then $y_{-i}^m < y_{-i}^n$, and apparently both values are positive since the intersection can only happen above the horizontal axis.

The comparative statics conclusions then follow in a direct way. Since any selection of $y^*(y_{-i})$ is strictly decreasing in y_{-i} before it hits zero, $y_i^n < y_i^m$. Since any selection of $z^*(y_{-i})$ is strictly decreasing in y_{-i} before it hits y_{-i}, $z^n < z^m$. Lastly, $V^n > V^m$ follows from the fact that $V(w, y_{-i})$ is strictly increasing in y_{-i} by the Envelop Theorem, which has been shown in the Proof of Proposition 2 (part 3). Q.E.D.

Proof of Corollary 6

In an n-player game, let y_1 denote the "monopoly contribution" amount, i.e., $y_1 \in y^*(0; w)$, and assume there exists an partially symmetric equilibrium where there are m contributors each donating y_m to the public good and the other $n - m$ players do not donate at all, $m \in [2, n]$. Then the welfare associated with the two equilibria are, respectively,

$$\begin{aligned} W_1 = & \; U(\frac{1}{p}(w - y_1), f(y_1)) + (n-1)U(\frac{w}{p}, f(y_1)) \\ W_m = & \; mU(\frac{1}{p}(w - y_m), f(my_m)) + (n-m)U(\frac{w}{p}, f(my_m)). \end{aligned}$$

Since m players each contributing y_m also constitutes a Nash equilibrium in an m-player game, by Proposition 5, $y_1 = z^1 > z^m = my_m$. Therefore, $U(\frac{w}{p}, f(y_1)) > U(\frac{w}{p}, f(my_m))$ as $U_2 > 0$ and $f' > 0$. Then $W_1 > W_m$ if

$$U(\frac{1}{p}(w - y_1), f(y_1)) + (m-1)U(\frac{w}{p}, f(y_1)) > mU(\frac{1}{p}(w - y_m), f(my_m)).$$

Indeed,

$$\begin{aligned} mU(\frac{1}{p}(w - y_m), f(my_m)) &< U(\frac{1}{p}(w - my_m), f(my_m)) + (m-1)U(\frac{w}{p}, f(my_m)) \\ &< U(\frac{1}{p}(w - y_1), f(y_1)) + (m-1)U(\frac{w}{p}, f(y_1)), \end{aligned}$$

the first inequality is from $U_{11} > 0$ while the second inequality is because $y_1 \in \arg\max_{y_i \in [0,w]} U(\frac{1}{p}(w - y_i), f(y_i))$ and $my_m < y_1 \leq w$, so $U(\frac{1}{p}(w - my_m), f(my_m)) \leq U(\frac{1}{p}(w - y_1), f(y_1))$. Q.E.D.